Symplectic Topology and Floer Homology

Volume 1

Published in two volumes, this is the first book to provide a thorough and systematic explanation of symplectic topology, and the analytical details and techniques used in applying the machinery arising from Floer theory as a whole.

Volume 1 covers the basic materials of Hamiltonian dynamics and symplectic geometry and the analytic foundations of Gromov’s pseudoholomorphic curve theory. One novel aspect of this treatment is the uniform treatment of both closed and open cases and a complete proof of the boundary regularity theorem of weak solutions of pseudoholomorphic curves with totally real boundary conditions. Volume 2 provides a comprehensive introduction to both Hamiltonian Floer theory and Lagrangian Floer theory, including many examples of their applications to various problems in symplectic topology.

*Symplectic Topology and Floer Homology* is a comprehensive resource suitable for experts and newcomers alike.

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Symplectic Topology and Floer Homology
Volume 1: Symplectic Geometry and Pseudoholomorphic Curves

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This is a two-volume series of monographs. This series provides a self-contained exposition of basic Floer homology in both open and closed string contexts, and systematic applications to problems in Hamiltonian dynamics and symplectic topology. The basic objects of study in these two volumes are the geometry of Lagrangian submanifolds and the dynamics of Hamiltonian diffeomorphisms and their interplay in symplectic topology.

The classical Darboux theorem in symplectic geometry reveals the flexibility of the group of symplectic transformations. On the other hand, Gromov and Eliashberg’s celebrated theorem (El87) reveals the subtle rigidity of symplectic transformations: the subgroup $\text{Symp}(M, \omega)$ consisting of symplectomorphisms is closed in $\text{Diff}(M)$ with respect to the $C^0$ topology. This demonstrates that the study of symplectic topology is subtle and interesting. Eliashberg’s theorem relies on a version of the non-squeezing theorem, such as the one proved by Gromov (Gr85) using the machinery of pseudoholomorphic curves. Besides Eliashberg’s original combinatorial proof of this non-squeezing result, there is another proof given by Ekeland and Hofer (EkH89) using the classical variational approach of Hamiltonian systems. The interplay between these two facets of symplectic geometry, namely the analysis of pseudoholomorphic curves and Hamiltonian dynamics, has been the main driving force in the development of symplectic topology since Floer’s pioneering work on his semi-infinite dimensional homology theory, which we now call Floer homology theory.

Hamilton’s equation $\dot{x} = X_H(t, x)$ arises in Hamiltonian mechanics and the study of its dynamics has been a fundamental theme of investigation in physics since the time of Lagrange, Hamilton, Jacobi, Poincaré and others. Many mathematical tools have been developed in the course of understanding its dynamics and finding explicit solutions of the equation. One crucial tool for the study of the questions is the least action principle: a solution of Hamilton’s equation
corresponds to a critical point of some action functional. In this variational principle, there are two most important boundary conditions considered for the equation $\dot{x} = X_H(t, x)$ on a general symplectic manifold: one is the \textit{periodic} boundary condition $\gamma(0) = \gamma(1)$, and the other is the \textit{Lagrangian} boundary condition $\gamma(0) \in L_0$, $\gamma(1) \in L_1$ for a given pair $(L_0, L_1)$ of two Lagrangian submanifolds. A submanifold $i : L \hookrightarrow (M, \omega)$ is called Lagrangian if $i^* \omega = 0$ and $\dim L = \frac{1}{2} \dim M$. This replaces the \textit{two-point} boundary condition in classical mechanics.

A diffeomorphism $\phi$ of a symplectic manifold $(M, \omega)$ is called a Hamiltonian diffeomorphism if $\phi$ is the time-one map of $\dot{x} = X_H(t, x)$ for some (time-dependent) Hamiltonian $H$. The set of such diffeomorphisms is denoted by $\text{Ham}(M, \omega)$. It forms a subgroup of $\text{Symp}(M, \omega)$. However, in the author’s opinion, it is purely a historical accident that Hamiltonian diffeomorphisms are studied because the definition of $\text{Ham}(M, \omega)$ is not a priori natural. For example, it is not a structure group of any geometric structure associated with the smooth manifold $M$ (or at least not of any structure known as yet), unlike the case of $\text{Symp}(M, \omega)$, which is the automorphism group of the symplectic structure $\omega$. Mathematicians’ interest in $\text{Ham}(M, \omega)$ is largely motivated by the celebrated conjecture of Arnol’d (Ar65), and Floer homology was invented by Floer (Fl88b, Fl89b) in his attempt to prove this conjecture.

Since the advent of Floer homology in the late 1980s, it has played a fundamental role in the development of symplectic topology. (There is also the parallel notion in lower-dimensional topology which is not touched upon in these two volumes. We recommend for interested readers Floer’s original article (Fl88c) and the masterpiece of Kronheimer and Mrowka (KM07) in this respect.) Owing to the many technicalities involved in its rigorous definition, especially in the case of Floer homology of Lagrangian intersections (or in the context of ‘open string’), the subject has been quite inaccessible to beginning graduate students and researchers coming from other areas of mathematics. This is partly because there is no existing literature that systematically explains the problems of symplectic topology, the analytical details and the techniques involved in applying the machinery embedded in the Floer theory as a whole. In the meantime, Fukaya’s categorification of Floer homology, i.e., his introduction of an $A_{\infty}$ category into symplectic geometry (now called the Fukaya category), and Kontsevich’s homological mirror symmetry proposal, followed by the development of open string theory of $D$ branes in physics, have greatly enhanced the Floer theory and attracted much attention from other mathematicians and physicists as well as the traditional symplectic geometers and topologists. In addition, there has also been considerable research into applications of symplectic ideas to various problems in (area-preserving) dynamical systems in two dimensions.
Our hope in writing these two volumes is to remedy the current difficulties to some extent. To achieve this goal, we focus more on the foundational materials of Floer theory and its applications to various problems arising in symplectic topology, with which the author is more familiar, and attempt to provide complete analytic details assuming the reader’s knowledge of basic elliptic theory of (first-order) partial differential equations, second-year graduate differential geometry and first-year algebraic topology. In addition, we also try to motivate various constructions appearing in Floer theory from the historical context of the classical Lagrange–Hamilton variational principle and Hamiltonian mechanics. The choice of topics included in the book is somewhat biased, partly due to the limitations of the author’s knowledge and confidence level, and also due to his attempt to avoid too much overlap with the existing literature on symplectic topology. We would like to particularly cite the following three monographs among others and compare these two volumes with them:


(There is another more recent monograph by Audin and Damian (AD14), which was originally written in French and then translated into English.)

First of all, Parts 2 and 3 of these two volumes could be regarded as the prerequisite for graduate students or post-docs to read the book (3) (FOOO09) in that the off-shell setting of Lagrangian Floer theory in Volume 2 presumes the presence of non-trivial instantons, or non-constant holomorphic discs or spheres. However, we largely limit ourselves to the monotone case and avoid the full-fledged obstruction–deformation theory of Floer homology which would inevitably involve the theory of $A_\infty$-structures and the abstract perturbation theory of virtual moduli technique such as the Kuranishi structure, which is beyond the scope of these two volumes. Luckily, the books (2) (Se08) and (3) (FOOO09) cover this important aspect of the theory, so we strongly encourage readers to consult them. We also largely avoid any extensive discussion on the Floer theory of exact Lagrangian submanifolds, except for the cotangent bundle case, because Seidel’s book (Se08) presents an extensive study of the Floer theory and the Fukaya category in the context of exact symplectic geometry to which we cannot add anything. There is much overlap of the materials in Part 2 on the basic pseudoholomorphic curve theory with Chapters 1–6 of the book by McDuff and Salamon (1) (MSa04). However, our exposition
of the materials is quite different from that of \((\text{MSa04})\). For example, from the beginning, we deal with pseudoholomorphic curves of arbitrary genus and with a boundary and unify the treatment of both closed and open cases, e.g., in the regularity theory of weak solutions and in the removal singularity theorem. Also we discuss the transversality issue after that of compactness, which seems to be more appropriate for accommodating the techniques of Kuranishi structure and abstract perturbation theory when the readers want to go beyond the semi-positive case. There are also two other points that we are particularly keen about in our exposition of pseudoholomorphic curve theory. One is to make the relevant geometric analysis resemble the style of the more standard geometric analysis in Riemannian geometry, emphasizing the tensor calculations via the canonical connection associated with the almost-Kähler property whenever possible. In this way we derive the relevant \(W^{k,p}\)-coercive estimates, especially an optimal \(W^{2,2}\)-estimate with Neumann boundary condition, by pure tensor calculations and an application of the Weitzenböck formula. The other is to make the deformation theory of pseudoholomorphic curves resemble that of holomorphic curves on (integrable) Kähler manifolds. We hope that this style of exposition will widen the readership beyond the traditional symplectic geometers to graduate students and researchers from other areas of mathematics and enable them to more easily access important developments in symplectic topology and related areas.

Now comes a brief outline of the contents of each part of the two volumes. The first volume consists of Parts 1 and 2.

Part 1 gives an introduction to symplectic geometry starting from the classical variational principle of Lagrange and Hamilton in classical mechanics and introduces the main concepts in symplectic geometry, i.e., Lagrangian submanifolds, Hamiltonian diffeomorphisms and symplectic fibrations. It also introduces Hofer’s geometry of Hamiltonian diffeomorphisms. Then the part ends with the proof of the Gromov–Eliashberg \(C^0\)-rigidity theorem \((\text{El87})\) and the introduction to continuous Hamiltonian dynamics and the concept of Hamiltonian homeomorphisms introduced by Müller and the present author \((\text{OhM07})\).

Part 2 provides a mostly self-contained exposition of the analysis of pseudoholomorphic curves and their moduli spaces. We attempt to provide the optimal form of a-priori elliptic estimates for the nonlinear Cauchy–Riemann operator \(\overline{\partial}_J\) \textit{in the off-shell setting}. For this purpose, we emphasize our usage of the canonical connection of the almost-Kähler manifold \((M,\omega,J)\). Another novelty of our treatment of the analysis is a complete proof of the boundary regularity theorem of weak solutions (in the sense of Ye \((\text{Ye94})\)) of \(J\)-holomorphic curves with totally real boundary conditions. As far as we
know, this regularity proof has not been given before in the existing literature. We also give a complete proof of compactness of the stable map moduli space following the approach taken by Fukaya and Ono (FOn99). The part ends with an explanation of how compactness–cobordism analysis of the moduli space of (perturbed) pseudoholomorphic curves combined with a bit of symplectic topological data give rise to the proofs of two basic theorems in symplectic topology; Gromov’s non-squeezing theorem and the nondegeneracy of Hofer’s norm on $\text{Ham}(M, \omega)$ (for tame symplectic manifolds).

The second volume consists of Parts 3 and 4. Part 3 gives an introduction to Lagrangian Floer homology restricted to the special cases of monotone Lagrangian submanifolds. It starts with an overview of Lagrangian intersection Floer homology on cotangent bundles and introduces all the main objects of study that enter into the recent Lagrangian intersection Floer theory without delving too much into the technical details. Then it explains the compactification of Floer moduli spaces, the details of which are often murky in the literature. The part ends with the construction of a spectral sequence, a study of Maslov class obstruction to displaceable Lagrangian submanifolds and Polterovich’s theorem on the Hofer diameter of $\text{Ham}(S^2)$.

Part 4 introduces Hamiltonian Floer homology and explains the complete construction of spectral invariants and various applications. The applications include construction of the spectral norm, Usher’s proofs of the minimality conjecture in the Hofer geometry and the optimal energy–capacity inequality. In particular, this part contains a complete self-contained exposition of the Entov–Polterovich construction of spectral quasimorphisms and the associated symplectic quasi-states. The part ends with further discussion of topological Hamiltonian flows and their relation to the geometry of area-preserving homeomorphisms in two dimensions.

The prerequisites for the reading of these two volumes vary part by part. A standard first-year graduate differentiable manifold course together with a little bit of knowledge on the theory of fiber bundles should be enough for Part 1. However, Part 2, the most technical part of the book, which deals with the general theory of pseudoholomorphic curves, the moduli spaces thereof and their stable map compactifications, assumes readers’ knowledge of the basic language of Riemannian geometry (e.g., that of Volumes 1 and 2 of Spivak (Spi79)), basic functional analysis (e.g., Sobolev embedding and Rellich compactness and others), elliptic (first-order) partial differential equations and first-year algebraic topology. The materials in Parts 3 and 4, which deal with the main topics of Floer homology both in the open and in the closed string context, rely on the materials of Parts 1 and 2 and should be readable on their own. Those who are already familiar with basic symplectic geometry
and analysis of pseudoholomorphic curves should be able to read Parts 3 and 4 immediately. This book can be used as a graduate textbook for the introduction to Gromov and Floer’s analytic approach to modern symplectic topology. Readers who would like to learn more about various deeper aspects of symplectic topology and mirror symmetry are strongly encouraged to read the books (1)–(3) mentioned above in addition, depending on their interest.

The author would like to end this preface by recalling his personal experience and perspective, which might not be shared by others, but which he hopes may help readers to see how the author came up with the current shape of these two volumes. The concept of symplectic topology emerged from Eliashberg and Gromov’s celebrated symplectic rigidity theorem. Eliashberg’s original proof was based on the existence of some $C^0$-type invariant of the symplectic diffeomorphism which measures the size of domains in the symplectic vector space. The existence of such an invariant was first established by Gromov as a corollary of his fundamental non-squeezing theorem that was proven by using the analytical method of pseudoholomorphic curves. With the advent of the method of pseudoholomorphic curves developed by Gromov and Floer’s subsequent invention of elliptic Morse theory that resulted in Floer homology, the landscape of symplectic geometry changed drastically. Many previously intractable problems in symplectic geometry were solved by the techniques of pseudoholomorphic curves, and the concept of symplectic topology gradually began to take shape.

There are two main factors determining how the author shaped the structure of the present book. The first concerns how the analytical materials are treated in Volume 1. The difficulty, or the excitement, associated with the method of pseudoholomorphic curves at the time of its appearance was that it involves a mathematical discipline of a nature very different from the type of mathematics employed by the mainstream symplectic geometers at that time. As a result the author feels that it created some discontinuity between the symplectic geometry before and after Gromov’s paper appeared, and the analysis presented was given quite differently from how such matters are normally treated by geometric analysts of Riemannian geometry. For example, the usage of tensor calculations is not emphasized as much as in Riemannian geometry. Besides, in the author’s personal experience, there were two stumbling blocks hindering getting into the Gromov–Witten–Floer theory as a graduate student and as a beginning researcher working in the area of symplectic topology. The first was the need to get rid of some phobia towards the abstract algebraic geometric materials like the Deligne–Mumford moduli space $\overline{M}_{g,n}$ of stable curves, and the other was the need to absorb the large amount of analytical materials that enter into the study of moduli spaces of pseudoholomorphic maps from
the original sources in the literature of the relevant mathematics whose details are often too sketchy. It turns out that many of these details are in some sense standard in the point of view of geometric analysts and can be treated in a more effective way using the standard tensorial methods of Riemannian geometry.

The second concerns how the Floer theory is presented in the book. In the author’s personal experience, it seems to be most effective to learn the Floer theory both in the closed and in the open string context simultaneously. Very often problems on the Hamiltonian dynamics are solved via the corresponding problems on the geometry of Lagrangian intersections. For this reason, the author presents the Floer theory of the closed and the open string context at the same time. While the technical analytic details of pseudoholomorphic curves are essentially the same for both closed and open string contexts, the relevant geometries of the moduli space of pseudoholomorphic curves are different for the closed case and the open case of Riemann surfaces. This difference makes the Floer theory of Lagrangian intersection very different from that of Hamiltonian fixed points.

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I learned most of the materials on basic symplectic geometry from Alan Weinstein as his graduate student in Berkeley. Especially, the majority of the materials in Part 1 of Volume 1 are based on the lecture notes I had taken for his year-long course on symplectic geometry in 1987–1988. One could easily see the widespread influence of his mathematics throughout Part 1. I would like to take this chance to sincerely thank him for his invaluable support and encouragement throughout my graduate study.

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List of conventions

We follow the conventions of (Oh05c, Oh09a, Oh10) for the definition of
Hamiltonian vector fields and action functionals and others appearing in the
Hamiltonian Floer theory and in the construction of spectral invariants and
Entov–Polterovich Calabi quasimorphisms. They are different from, e.g., those
used in (EnP03, EnP06, EnP09) in one way or another.

(1) The canonical symplectic form \( \omega_0 \) on the cotangent bundle \( T^*N \) is given by

\[
\omega_0 = -d\Theta = \sum_{i=1}^{n} dq_i \wedge dp_i,
\]
where $\Theta$ is the Liouville one-form given by $\Theta = \sum_{i=1}^{n} p_i \, dq^i$ in the canonical coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ associated with a coordinate system $(q^1, \ldots, q^n)$ on $N$.

(2) The Hamiltonian vector field $X_H$ on a symplectic manifold $(M, \omega)$ is defined by $dH = \omega(X_H, \cdot)$.

(3) The action functional $\mathcal{A}_H : \mathcal{L}_0(M) \to \mathbb{R}$ is defined by

$$\mathcal{A}_H([\gamma, w]) = - \int w^* \omega - \int_0^1 H(t, \gamma(t)) \, dt.$$  

(4) In particular, $\mathcal{A}_H$ is reduced to the classical Hamilton action functional on the path space of $T^*N$,

$$\int_{\gamma} p \, dq - \int_0^1 H(t, \gamma(t)) \, dt,$$

which coincides with the standard definition in the literature of classical mechanics.

(5) An almost-complex structure is called $J$-positive if $\omega(X, JX) \geq 0$ and $J$-compatible if the bilinear form $\omega(\cdot, J\cdot)$ defines a Riemannian metric.

(6) Note that $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ carries three canonical bilinear forms: the symplectic form $\omega_0$, the Euclidean inner product $g$ and the Hermitian inner product $\langle \cdot, \cdot \rangle$. We take the Hermitian inner product to be complex linear in the first argument and anti-complex linear in the second argument. Our convention for the relation among these three is

$$\langle \cdot, \cdot \rangle = g(\cdot, \cdot) - i\omega_0(\cdot, \cdot).$$

**Comparison with the Entov–Polterovich convention (EnP09)**

In (EnP03, EnP06, EnP09), Entov and Polterovich used different sign conventions from the ones in (Oh05c) and the present book. If we compare our convention with the one from (EnP09), the following is the list of differences.

(1) The canonical symplectic form on $T^*N$ in their convention is

$$\overline{\omega}_0 := d\Theta = \sum_{i=1}^{n} dp_i \wedge dq^i.$$  

(2) The definition of the Hamiltonian vector field in (EnP09) is

$$dH = \omega(\cdot, X_H).$$
Therefore, by replacing $H$ by $-H$, one has the same set of closed loops as the periodic solutions of the corresponding Hamiltonian vector fields on a given symplectic manifold $(M, \omega)$ in both conventions.

(3) Combination of (1) and (2) makes the Hamiltonian vector field associated with a function $H = H(t, q, p)$ on the cotangent bundle give rise to the same vector field. In particular, the classical Hamiltonian vector field on the phase space $\mathbb{R}^{2n}$ with canonical coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ is given by the expression

$$X_H = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right).$$

(4) For the definition of the action functional (EnP03) and (EnP09) take

$$- \int w^* \omega + \int_0^1 H(t, \gamma(t)) dt. \quad (0.0.1)$$

We denote the definition (0.0.1) by $\overline{A}_H([\gamma, w])$ for the purpose of comparison of the two below.

(5) In particular, $\overline{A}_H$ is reduced to

$$- \int_\gamma p \, dq + \int_0^1 H(t, \gamma(t)) dt$$

on the path space of $T^*N$, which is the negative of the standard definition of Hamilton’s action functional in the literature of classical mechanics (e.g., (Ar89) and (Go80)).

(6) Since these two conventions use the same associated almost-Kähler metric $\omega(\cdot, J\cdot)$, the associated perturbed Cauchy–Riemann equations have exactly the same form; in particular, they have the same sign in front of the Hamiltonian vector field.

(7) In addition, Entov and Polterovich (EnP03, EnP06) use the notation $c(a, H)$ for the spectral numbers, where $a$ is the quantum homology class, while we use $a$ to denote a quantum cohomology class. The comparison is the following:

$$\rho(H; a) = c(a^\flat; \overline{H}) = c(a^\flat; H), \quad (0.0.2)$$

where $a^\flat$ is the homology class dual to the cohomology class $a$ and $\overline{H}$ is the inverse Hamiltonian of $H$ given by

$$\overline{H}(t, x) = -H(t, \phi_H^t(x)). \quad (0.0.3)$$

(8) The relationship among the three bilinear forms on $\mathbb{R}^{2n}$ is given by

$$\langle \cdot, \cdot \rangle = g(\cdot, \cdot) + i\tilde{\omega}_0(\cdot, \cdot),$$
for which the inner product is complex linear in the second argument and anti-complex linear in the first argument.

With these understood, one can translate every statement in (EnP03, EnP06) into ones in terms of our notations.
PART 1

Hamiltonian dynamics and symplectic geometry
The least action principle and Hamiltonian mechanics

In this chapter, we start by reviewing Hamilton’s least action principle in classical mechanics. We will motivate all the basic concepts in symplectic geometry out of this variational principle. We refer the reader to (Go80) or (Ar89) for further physical applications of this principle.

1.1 The Lagrangian action functional and its first variation

We start with our explanations on \( \mathbb{R}^n \) and then move onto general configuration space \( M \). We call \( \mathbb{R}^n \) the (configuration) space and \( \mathbb{R} \times \mathbb{R}^n \) the space-time. We denote by \( q^i, i = 1, \ldots, n \) the standard coordinate functions of \( \mathbb{R}^n \), and by

\[
(q^1, \ldots, q^n, v^1, \ldots, v^n)
\]

the associated canonical coordinates of \( T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \). It is customary to denote the canonical coordinates by

\[
(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)
\]

instead, especially in the physics literature. We will follow this convention, whenever there is no danger of confusion.

We denote by \( \gamma : [t_0, t_1] \to \mathbb{R}^n \) a continuous path, regarded as the trajectory of a moving particle. In coordinates, we may write

\[
\gamma(t) = (q^1(t), \ldots, q^n(t)), \quad \dot{q}^1(t) = \dot{q}^i(\gamma(t)).
\]

We say \([t_0, t_1]\) is the domain of the path and denote

\[
\mathcal{P} = \mathcal{P}_{t_0}^{t_1}(\mathbb{R}^n) = \{ \gamma \mid \text{Dom}(\gamma) = [t_0, t_1] \}.
\]
In the Lagrangian formalism of classical mechanics, the relevant action functional, called the *Lagrangian action functional*, has the form

\[ \Phi(\gamma) = \int_{t_0}^{t_1} L(t, \gamma, \dot{\gamma}) dt \]

where \( L \) is a function, called the *Lagrangian*,

\[ L = L(t, q, \dot{q}) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}. \]

**Example 1.1.1** Consider a motion on \( \mathbb{R}^n \) over the time interval \([t_0, t_1]\), i.e., a map \( \gamma : [t_0, t_1] \to \mathbb{R}^n \).

(i) The *energy functional* is defined by

\[ E(\gamma) = \frac{1}{2} \int_{t_0}^{t_1} |\dot{\gamma}|^2 dt. \]

(ii) The length of the path \( \gamma : [t_0, t_1] \to \mathbb{R}^n \) is given by

\[ L(\gamma) = \int_{t_0}^{t_1} |\dot{\gamma}| dt. \]

As in the mechanics literature, we denote by \( \Delta \gamma \) the *infinitesimal variation*. In the more formal presentation, we note that the tangent space of \( \mathbb{R}^n \) at a given point \( x \) is canonically identified with \( \mathbb{R}^n \) itself. Therefore we can write a variation \( h = \Delta \gamma \in T_{\gamma} P \), the tangent space, at the path \( \gamma \) as a map

\[ h : [t_0, t_1] \to \mathbb{R}^n. \]

We denote by \(|h|\) the norm of \( h \) with respect to a given norm on the linear space

\[ T_{\gamma} P_{t_0} (\mathbb{R}^n) \cong \Gamma(\gamma^*(T\mathbb{R}^n)). \]

We will not delve into the matter of giving the precise mathematical description of the following definition, which is the analog of the standard definitions to the finite dimensional case in the present *infinite-dimensional* case. We refer to e.g., (AMR88) for the precise mathematical definitions.

**Definition 1.1.2** Let \( \Phi \) be as above.

1. A functional \( \Phi \) is *differentiable* at \( x \) if there exists a linear map \( F(\gamma) \) such that

\[ \Phi(\gamma + \Delta \gamma) - \Phi(\gamma) = F(\gamma) \cdot \Delta \gamma + R, \quad (1.1.1) \]

where \( R = R(\gamma, h) = O(|h|^2) \).
(2) If this holds, \( F(\gamma) \) is said to be the differential of \( \Phi \) at \( \gamma \) and denoted by 
\[ d\Phi(\gamma). \]
(3) We call any path \( \gamma \) an extremal path if it satisfies \( F(\gamma) \cdot h = 0 \) for all variation \( h \).

Now we find the formula for the differential \( F(\gamma) \cdot h \) in terms of the Lagrangian density \( L \). For given \( h : [t_0, t_1] \to \mathbb{R}^n \) regarded as a variation (or a tangent vector) of \( \gamma \), we have
\[
F(\gamma) \cdot h = \left. \frac{d}{ds} \right|_{s=0} \Phi(\gamma + sh)
\]
\[
= \left. \frac{d}{ds} \right|_{s=0} \int_{t_0}^{t_1} L(t, q + sh, \dot{q} + s\dot{h}) dt
\]
\[
= \int_{t_0}^{t_1} \frac{d}{ds} \left. L(t, q + sh, \dot{q} + s\dot{h}) \right|_{s=0} dt
\]
\[
= \int_{t_0}^{t_1} \left( \frac{\partial L}{\partial q} \cdot h + \frac{\partial L}{\partial \dot{q}} \cdot \dot{h} \right) dt.
\]
We integrate the second term by parts and get
\[
\int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}} \cdot \dot{h} dt = \left. \frac{\partial L}{\partial \dot{q}} \cdot h \right|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \cdot h dt.
\]
Therefore,
\[
F(\gamma) \cdot h = \int_{t_0}^{t_1} \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \right) \cdot h dt + \left. \frac{\partial L}{\partial \dot{q}} \cdot h \right|_{t_0}^{t_1}. \tag{1.1.2}
\]
To describe the condition of extremal paths of \( \Phi \) in terms of the Lagrangian density function \( L \), we require that the boundary term, i.e., the second term of (1.1.2), vanish. There are two common ways of achieving this goal in the mechanics.

1. **Two-point boundary condition.** We define the subset
\[
\mathcal{P}_{t_0}^{t_1}(\mathbb{R}^n; q_0, q_1) = \{ \gamma \in \mathcal{P}_{t_0}^{t_1}(\mathbb{R}^n) | q(t_0) = q_0, \quad q(t_1) = q_1 \} \tag{1.1.3}
\]
of \( \mathcal{P}_{t_0}^{t_1}(\mathbb{R}^n) \), which consists of the paths satisfying the so-called two-point boundary condition. In this case, the variation \( h = \Delta \gamma \) satisfies
\[
h(t_0) = 0 = h(t_1).
\]
For such \( \gamma \) and \( h \), we have
\[
\left. \frac{\partial L}{\partial \dot{q}} \cdot h \right|_{t_0}^{t_1} = \frac{\partial L}{\partial \dot{q}}(t_1, \gamma(t_1), \dot{\gamma}(t_1)) \cdot h(t_1) - \frac{\partial L}{\partial \dot{q}}(t_0, \gamma(t_0), \dot{\gamma}(t_0)) \cdot h(t_0) = 0 - 0 = 0.
\]
The least action principle and Hamiltonian mechanics

Therefore, if we restrict $\Phi$ to this subset $P^t_{t_0}(\mathbb{R}^n; q_0, q_1)$, the corresponding restricted functional has the first variation given by

$$F(\gamma) \cdot h = \int_{t_0}^{t_1} \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \right) \cdot h \, dt.$$  \hspace{1cm} (1.1.4)

Hence we have derived the equation of motion.

**Proposition 1.1.3** Let $q_0$ and $q_1$ be two fixed points in $\mathbb{R}^n$. Consider the functional

$$\Phi(\gamma) = \int_{t_0}^{t_1} L(t, \gamma, \dot{\gamma}) \, dt$$

for the paths $\gamma$ under the two-point boundary condition

$$\gamma(0) = q_0, \quad \gamma(1) = q_1.$$

Then $\gamma$ is extremal if and only if it satisfies

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) = 0, \quad i = 1, \ldots, n.$$  \hspace{1cm} (1.1.5)

We next discuss another natural boundary condition, the periodic boundary condition.

**2. Periodic boundary conditions.** Consider the subset

$$L^t_{t_0}(\mathbb{R}^n) = \{ \gamma \in P^t_{t_0}(\mathbb{R}^n) \mid \gamma(t_0) = \gamma(t_1), \dot{\gamma}(t_0) = \dot{\gamma}(t_1) \}.$$

The corresponding variation $h = \Delta$ will satisfy the periodic boundary condition

$$h(t_0) = h(t_1).$$

In addition, provided the density function $L = L(t, q, \dot{q})$ satisfies the time-periodic condition

$$L(t_0, \cdot, \cdot) \equiv L(t_1, \cdot, \cdot),$$  \hspace{1cm} (1.1.6)

the boundary term of (1.1.2) again vanishes; this time, however, because we have

$$\frac{\partial L}{\partial \dot{q}}(t_1, \gamma(t_1), \dot{\gamma}(t_1)) \cdot h(t_1) = \frac{\partial L}{\partial \dot{q}}(t_0, \gamma(t_0), \dot{\gamma}(t_0)) \cdot h(t_0).$$

We summarize this as follows.

**Proposition 1.1.4** Suppose $L$ satisfies (1.1.6). Consider the functional

$$\Phi(\gamma) = \int_{t_0}^{t_1} L(t, \gamma, \dot{\gamma}) \, dt$$
restricted to the paths $\gamma$ under the periodic boundary condition

$$\gamma(t_0) = \gamma(t_1), \quad \dot{\gamma}(t_0) = \dot{\gamma}(t_1).$$

Then $\gamma$ is extremal if and only if it satisfies (1.1.5).

Equation (1.1.5) is called the Euler–Lagrange equation of $L$. Note that this is the second-order ODE with respect to the variable $\gamma(t) = (q^1(t), \ldots, q^n(t))$.

**Remark 1.1.5** Since the above discussion is independent of the choice of coordinates $(q^1, \ldots, q^n)$, as long as we use the associated canonical coordinates of $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, the Euler–Lagrange equation for $L$ is also coordinate-independent (or covariant in the physics terminology). In other words, if $(Q^1, \ldots, Q^n)$ is another coordinate system of $\mathbb{R}^n$, then the associated Euler–Lagrange equation has the same form as

$$\frac{\partial L}{\partial Q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{Q}_i} \right) = 0, \quad i = 1, \ldots, n.$$

### 1.2 Hamilton’s action principle

The Lagrangian that is relevant in Newtonian mechanics has the form

$$L = T - U, \quad (1.2.7)$$

where $T$ is the kinetic energy

$$T = T(x, \dot{x}) = \frac{1}{2} m \dot{x} \cdot \dot{x}$$

and $U = U(x) : \mathbb{R}^n \to \mathbb{R}$ is the potential energy, which is a function depending only on the position vector $x \in \mathbb{R}^n$. Then Newton’s equation of motion

$$\frac{d}{dt} (m\dot{q}^i) = - \frac{\partial U}{\partial q^i}, \quad i = 1, \ldots, n, \quad (1.2.8)$$

is equivalent to the Euler–Lagrange equation (1.1.5) for the Lagrangian (1.2.7). This gives rise to the following:

**Hamilton’s least action principle.** Motions of the mechanical system under Newton’s second law coincide with the extremals of the functional

$$\Phi(\gamma) = \int_{t_0}^{t_1} L \, dt$$

(under an appropriate boundary condition as mentioned before).
Recall that, in Newtonian mechanics, the momentum vector is defined by

\[ m \dot{x} = p \quad (1.2.9) \]

and the force field is provided by

\[ -\nabla U = F. \]

Owing to the special form of the Lagrangian given in (1.2.7), \( p_i = m \dot{q}^i \) is nothing but \( \partial L / \partial \dot{q}^i \). This leads to the following notion in classical mechanics. (See (Go80) or (Ar89) for more discussion.)

**Definition 1.2.1** Let \( L \) be an arbitrary Lagrangian. Then the generalized momenta, denoted as \( p_i \), are defined by

\[ p_i = \frac{\partial L}{\partial \dot{q}^i}. \quad (1.2.10) \]

With this definition, the Euler–Lagrange equation becomes

\[ \dot{p}_i = \frac{\partial L}{\partial q^i}, \quad i = 1, \ldots, n. \quad (1.2.11) \]

**Exercise 1.2.2** Interpret (1.2.10) and (1.2.11) in the invariant fashion. Explain why one should regard the right-hand side thereof as a covariant one-tensor or as a differential one-form.

### 1.3 The Legendre transform

In solving a mechanical problem, one often first finds the formula for the momenta \( p_i \) in time and then would like to convert this into a formula for the position coordinates \( q^i \). This is not always possible, though. A necessary condition for the Lagrangian \( L \) is its convexity with respect to \( \dot{x} \) for each fixed position vector \( x \). Such a function should be considered as a family of convex functions

\[ \dot{x} \mapsto L(t, x, \dot{x}); \quad \mathbb{R}^n \to \mathbb{R} \]

parameterized by the position vector \((t, x) \in \mathbb{R} \times \mathbb{R}^n\).

In this section, we discuss an important operation, called the Legendre transform, that appears in many branches of mathematics. The Legendre transform recently played an important role in a rigorous formulation of the mirror symmetry in relation to the Strominger–Yau–Zaslow proposal. We refer the reader to (SYZ01), (Hi99) and (GrSi03) for more explanation of this aspect. Partly
because of this recent resurgence of interest, we provide some detailed mathematical explanations of the Legendre transform in an invariant fashion. After that, we will return to the Hamiltonian formulation of the classical mechanics.

1.3.1 The Legendre transform of a function

Let $V$ be a (finite-dimensional) vector space and $V^*$ its dual vector space. We denote by $\langle \cdot, \cdot \rangle$ the canonical paring between $V$ and $V^*$.

**Definition 1.3.1** Let $U \subset V$ be an open subset. A function $f : V \to \mathbb{R}$ is said to be **convex** on $U \subset V$ if it satisfies

$$f((1-t)x_1 + tx_2) \leq (1-t)f(x_1) + tf(x_2) \quad (1.3.12)$$

for all $t \in [0,1]$ and for all $x_1, x_2 \in U$, and **strictly convex** if

$$f((1-t)x_1 + tx_2) < (1-t)f(x_1) + tf(x_2) \quad (1.3.13)$$

for all $t \in (0,1)$ and for all $x_1, x_2 \in U$.

The following is an easy exercise to prove.

**Lemma 1.3.2**

1. Any convex function $f$ on $U$ is continuous on $U$.
2. Any strictly convex function $f : V \to \mathbb{R}$ that is bounded below has the unique minimum point if it has one.

**Example 1.3.3** Let $V = \mathbb{R}$ and consider the function $f(x) = x^\alpha/\alpha$ with $\alpha > 1$. Then $f$ is convex on $\mathbb{R}$.

For a given function $f : V \to \mathbb{R}$, we consider the function $F : V \times V^* \to \mathbb{R}$ defined by

$$F(x, p) := \langle x, p \rangle - f(x)$$

and the value

$$g(p) = \sup_{x \in V} F(x, p). \quad (1.3.14)$$

The new function $g$, if defined, is called the **Legendre transform** or the **Fenchel transform**.

In general, the value of $g$ need not be finite. However, whenever the value is defined, we have the inequality

$$\langle x, p \rangle \leq f(x) + g(p), \quad (1.3.15)$$
which is called the *Fenchel inequality*. To make the value of $g$ finite everywhere, one needs to impose the following superlinearity of $f$.

**Definition 1.3.4** Let $V$ be a (finite-dimensional) vector space. A function $f : V \rightarrow \mathbb{R}$ is said to be superlinear, if $f$ is bounded below and

$$
\lim_{|x| \to \infty} \frac{f(x)}{|x|} = +\infty.
$$

(1.3.16)

**Exercise 1.3.5** Prove that the superlinearity in this definition is equivalent to the statement that, for all $K < \infty$, there exists $C(K) > -\infty$ such that $f(x) \geq K|x| + C(K)$ for all $x \in V$.

An example of a convex but not superlinear function is $f(x) = e^{-x}$ as a function on $V = \mathbb{R}$.

We borrow the following from Proposition 1.3.5 of Fathi’s book (*Fa05*) restricted to the finite-dimensional cases.

**Proposition 1.3.6** Let $f : V \rightarrow \mathbb{R}$ be a function. Then the following apply.

1. If $f$ is superlinear, then $g$ is finite everywhere.
2. If $g$ is finite everywhere, it is convex.
3. If $f$ is continuous, then $g$ is superlinear.

**Proof** We first prove (1). By the superlinearity (1.3.16) and Exercise 1.3.5, there exists a constant $C = C(|p|)$ such that $f(x) \geq |p||x| + C(|p|)$ for all $x \in V$. Therefore we have

$$
\langle x, p \rangle - f(x) \leq |x||p| - f(x) \leq |x||p| - (|x||p| + C(|p|)) = -C(|p|) < \infty.
$$

This proves $g(p) = \sup_{x \in V} (\langle x, p \rangle - f(x)) < \infty$. On the other hand, we have

$$
g(p) = \sup_{x \in V} (\langle x, p \rangle - f(x)) \geq -f(0) > -\infty,
$$

which proves (1). For the property (2), we note that $g$ is the upper bound for the family of linear functions, which is obviously convex,

$$
p \mapsto \langle x, p \rangle - f(x),
$$

and hence $g$ must be convex. To show (3), we will apply Exercise 1.3.5. For any $K$, we derive

$$
g(p) \geq \sup_{|x|=K} \langle x, p \rangle - \sup_{|x|=K} f(x)
$$
for all $K$. But we have $\sup_{|x|=K} \langle x, p \rangle = K|p|$ and $\sup_{|x|=K} f(x)$ is bounded since $f$ is assumed to be continuous. Note that the sphere $\{x \in V \mid |x| = K\}$ is compact because we assume $V$ is finite-dimensional. This finishes the proof. □

The following theorem is due to Fenchel. We leave its proof as an exercise or refer the reader to (Fa05).

**Theorem 1.3.7** (Fenchel) Suppose that $f : V \to \mathbb{R}$ is continuous and superlinear on $V$.

(1) If $f$ is convex and differentiable, then we have

$$\forall x \in V, \ g \circ df(x) = df(x)(x) - f(x).$$

Moreover $\langle x, p \rangle = g(p) + f(x)$ if and only if $p = df(x)$.

(2) We have $f(x) = \sup_{p \in V^*} (\langle x, p \rangle - g(p))$ if and only if $f$ is convex.

**Exercise 1.3.8** Prove Fenchel’s lemma, Theorem 1.3.7.

**Proposition 1.3.9** Suppose that $f : V \to \mathbb{R}$ is continuous and superlinear on a finite-dimensional vector space $V$, and let $g : V^* \to \mathbb{R}$ be its Legendre transform. Then the following hold.

(1) $g$ is everywhere continuous.

(2) For every $p \in V^*$, there exists $v \in V$ such that $\langle v, p \rangle = f(v) + g(p)$.

(3) If $f$ is convex, for every $x \in V$, there exists $p \in V^*$ such that $\langle x, p \rangle = f(x) + g(p)$.

**Proof** Statement (1) follows from the convexity of $g$. For the proof of (2), we first note that

$$|\langle x, p \rangle| \leq |x||p|, \ \lim_{|x| \to \infty} \frac{f(x)}{|x|} = \infty.$$ 

From this, we see that

$$\lim_{|x| \to \infty} \frac{\langle x, p \rangle - f(x)}{|x|} = -\infty.$$ 

Hence the supremum $g(p)$ of the continuous function $p \mapsto \langle x, p \rangle - f(x)$ is the same as the supremum of its restriction to a big enough bounded set. Therefore the supremum of $g$ is achieved. In other words, there exists $v \in V$ such that

$$g(p) = \langle v, p \rangle - f(v),$$
which finishes the proof. For (3), we remark that \((V^*)^* = V\) and \(f\) is the Legendre transform of \(g\) by Theorem 1.3.7 (2). Therefore it follows from (2).

\[\square\]

**Corollary 1.3.10** (Surjectivity) If \(f\) is convex, everywhere differentiable and superlinear, then \(df : V \to V^*\) is surjective.

**Proof** This follows from Theorem 1.3.7 (1) together with Proposition 1.3.9 (2).

The map \(L : x \mapsto df(x) ; V \to V^*\) is called the Legendre transform map associated with the function \(f\). The following is a re-statement of Theorem 1.3.7.

**Proposition 1.3.11** Let \(f : V \to \mathbb{R}\) be \(C^1\) and superlinear and \(L : V \to V^*\) be its Legendre transform map. If \(g : V^* \to \mathbb{R}\) is the Legendre transform of \(f\), then

\[\forall x \in V, \quad g \circ L(x) = df(x)(x) - f(x). \tag{1.3.17}\]

Moreover \(\langle x, p \rangle = f(x) + g(p)\) if and only if \(p = df(x)\).

**Theorem 1.3.12** (Injectivity) Suppose that \(f : V \to \mathbb{R}\) is a \(C^1\) convex function. Then its associated Legendre transform map \(L : V \to V^*\) is injective if and only if \(f\) is strictly convex.

**Proof** Let \(p \in V^*\) and \(F : V \times V^* \to \mathbb{R}\) be the function \(F(x, p) = \langle x, p \rangle - f(x)\). Then we have \(p = df(x)\) if and only if \(dF_p(x) = 0\). Therefore \(x\) is the unique maximum point of the strictly concave function \(F_p : x \mapsto \langle x, p \rangle - f(x)\). Hence \(L\) is injective.

Conversely, if \(L\) is injective, the concave function

\[x \mapsto \langle x, df(v) \rangle - f(x)\]

has only \(v\) as a critical point, so \(v\) must be the unique maximum point. Let \(v = (1 - t)x + ty\) with \(t \in (0, 1)\) with \(x \neq v \neq y\). Then we have

\[\langle df(v), x \rangle - f(x) < \langle df(v), v \rangle - f(v),\]

\[\langle df(v), y \rangle - f(y) < \langle df(v), v \rangle - f(v).\]

Since \(t, 1 - t > 0\), on taking the convex combination of these two inequalities and multiplying by \((-1)\) we obtain

\[(1 - t)f(x) + tf(y) - df(v)((1 - t)x + ty) > \langle df(v), v \rangle - f(v).\]
But we have chosen $x, y$ so that $(1 - t)x + ty = v$, so this implies $(1 - t)f(x) + tf(y) > f(v)$. This proves the strict convexity of $f$. □

On combining Corollary 1.3.10 and Theorem 1.3.12, we have the following.

**Corollary 1.3.13 (Bijectivity)** Suppose that $f : V \to \mathbb{R}$ is a $C^1$ convex function. Then its associated Legendre transform map $L : V \to V^*$ is bijective if and only if $f$ is strictly convex and superlinear. In that case, $L$ is a homeomorphism.

One can also easily see that, if $f$ is $C^2$ and superlinear, then $L$ is a $C^1$ diffeomorphism.

The following is an amusing example from (Ar89), which shows that Young’s inequality in the classical analysis is a special case of Fenchel’s inequality (1.3.15).

**Example 1.3.14** Consider the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$f(v) = \frac{v^\alpha}{\alpha}, \quad \alpha > 1.$$  

This function is $C^1$, strictly convex and superlinear, so its Legendre transform $g : \mathbb{R} \to \mathbb{R}$ is everywhere defined, which is again strictly convex and superlinear. In fact, one can apply

$$g(p) = \langle v, p \rangle - f(v), \quad p = \frac{df}{dv}(v), \quad (1.3.18)$$

to find the explicit formula for $g$. We compute $df/dv = x^{\alpha-1}$. Solving the equation $p = x^{\alpha-1}$ with respect to $x$ for any given $p$, we obtain

$$v = p^{1/(\alpha-1)}.$$  

Substituting this into (1.3.18), we derive

$$g(p) = p^{1+1/(\alpha-1)} - \frac{p^{\alpha/(\alpha-1)}}{\alpha} = \frac{p^\beta}{\beta},$$

where $\beta > 1$ is the unique positive number solving

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$  

Therefore, in this case, the Fenchel inequality (1.3.15) is reduced to Young’s inequality

$$xp \leq \frac{x^\alpha}{\alpha} + \frac{p^\beta}{\beta}$$

for any $x, p > 0$. 

1.3.2 The Legendre transform and the action functional

Now we consider the product $V \times V$, regarding it as the tangent bundle $TV$ of $V$. We denote by $x$ a point in $V$. Choose any coordinates $(q^1, \ldots, q^n)$. We denote by

$$(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$$

the associated canonical coordinates of $V \times V \cong TV$. We recall that the canonical coordinates are linear in the fiber direction of $TV$ with respect to the basis

$$\left\{ \frac{\partial}{\partial q^1} \bigg|_x, \ldots, \frac{\partial}{\partial q^n} \bigg|_x \right\}$$

at any given point $x \in V$. Therefore we can canonically talk about the fiberwise convexity of any given function $L : V \times V \to \mathbb{R}$ with respect to the fiber coordinates $(\dot{q}^1, \ldots, \dot{q}^n)$ at any given $x \in V$. Similarly we also consider the canonical coordinates of $V \times V^* = T^*V$,

$$(q^1, \ldots, q^n, p_1, \ldots, p_n).$$

We also consider a time-dependent family $L = L(t, x, \dot{x}) : \mathbb{R} \times TV \to \mathbb{R}$. We denote the (fiberwise) Legendre transform map of $L$ by $H : \mathbb{R} \times V \times V^* \to \mathbb{R}$ defined by

$$H(t, x, p) = \sup_{\dot{x} \in \mathbb{R}} (\langle \dot{x}, p \rangle - L(t, x, \dot{x})). \quad (1.3.19)$$

The function $H$ is called a Hamiltonian or a Hamiltonian function associated with the Lagrangian $L$.

Assuming that $L$ is (fiberwise) $C^1$ strictly convex and superlinear, the fiberwise Legendre transform map

$$\mathcal{L} : \mathbb{R} \times TV \to \mathbb{R} \times T^*V$$

defines a fiber-preserving bijective homeomorphism. Furthermore, we have the identity

$$H(t, x, p) = \langle v, p \rangle - L(t, x, v), \quad (1.3.20)$$

where $v = v(t, x, p) \in V$ is the unique solution solving the equation

$$p = d_x L(t, x, v). \quad (1.3.21)$$

In the canonical coordinates, we have

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad i = 1, \ldots, n,$$
1.3 The Legendre transform

which are exactly the generalized momenta associated with the Lagrangian $L$ as defined in Definition 1.2.1. Then the Euler–Lagrange equation becomes

$p_i = \frac{\partial L}{\partial \dot{q}_i}.$

Now we derive a fundamental theorem in the Hamiltonian mechanics. First we introduce the notion of prolongation.

**Definition 1.3.15** For a given path $\gamma = \gamma(t)$, we define its prolongation, denoted by $\tilde{\gamma} : \mathbb{R} \times V \times V^*$ by

$$\tilde{\gamma}(t) = (\gamma(t), p(t))$$

where

$$p(t) = d\dot{x}L(\gamma(t), \dot{\gamma}(t)).$$

We call this $p$ the **generalized momentum of $L$**.

Let $H$ be the Legendre transform of $L$.

We compute the differential of $H$ on $\mathbb{R} \times V \times V^*$ in two different ways. First, we have

$$dH = \frac{\partial H}{\partial t} dt + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q^j} dq^j. \quad (1.3.22)$$

On the other hand, by differentiating (1.3.20) considering $v$ as a function of $(t, x, p)$, we derive

$$dH = v^i dp_i + d\dot{v}^j p_i - \left( \frac{\partial L}{\partial t} dt + \frac{\partial L}{\partial q^j} dq^j + \frac{\partial L}{\partial \dot{q}^j} dv^j \right)$$

$$= v^i dp_i - \frac{\partial L}{\partial t} dt - \frac{\partial L}{\partial q^j} dq^j + \left( p_i - \frac{\partial L}{\partial \dot{q}^j} \right) d\dot{v}^j.$$

The last term drops out here because $v$ is the solution of $p = d\dot{x}L$, i.e., $p_i = \partial L/\partial \dot{q}_i$, $i = 1, \ldots, n$. Hence this equation is reduced to

$$dH = -\frac{\partial L}{\partial \dot{t}} dt + \dot{v}^i dp_i - \frac{\partial L}{\partial \dot{q}^j} dq^j. \quad (1.3.23)$$

Comparing (1.3.22) and (1.3.23), we have obtained

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial \dot{t}}, \quad \frac{\partial H}{\partial p_i} = \dot{v}^i, \quad \frac{\partial H}{\partial q^j} = -\frac{\partial L}{\partial \dot{q}^j}.$$

**Theorem 1.3.16** Let $H$ be the Legendre transform of $L$. Then a path $\gamma = \gamma(t)$ is a solution of the Euler–Lagrange equation on $V$ (the configuration space) if and only if its prolongation $\tilde{\gamma} : \mathbb{R} \times V \times V^*$ satisfies the following equation on $T^*V$ (the phase space)

$$\dot{q}^j = \frac{\partial H}{\partial p_i}, \quad \dot{p}_j = -\frac{\partial H}{\partial q^j}. \quad (1.3.24)$$
Proof Suppose that a path $x = x(t)$ is a solution of the Euler–Lagrange equation

$$ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0. $$

Then the prolongation $t \mapsto (x(t), p(t))$ with $p(t)$ given by $p_i = \frac{\partial L}{\partial \dot{q}^i}$ will satisfy

$$ \dot{p}_j = -\frac{\partial H}{\partial q^j}, \quad \dot{q}^i(= v^i) = \frac{\partial H}{\partial p_i}. $$

The converse will also follow on reading the above proof backwards. □

Equation (1.3.24) is called Hamilton’s equation associated with the function $H$. At this stage, we notice that Hamilton’s equation can be considered for an arbitrary function $H : \mathbb{R} \times V \times V^*$, not necessarily coming from the Legendre transform of a Lagrangian $L$. Theorem 1.3.16 then implies that Hamilton’s equation is reduced to the Euler–Lagrange equation when applied to $H$ that arises via the Legendre transform of a Lagrangian $L$. We rephrase Theorem 1.3.16 into the following invariant formulation.

**Theorem 1.3.17** Let $L$ be a strictly convex superlinear $C^2$ function and $\mathcal{L} : TV \to T^*V$ be the associated Legendre transform map. Then a path $\gamma : \mathbb{R} \to V$ is a solution of the Euler–Lagrange equation of $L$ if and only if $\mathcal{L}(\gamma)$ is a solution of Hamilton’s equation associated with the Legendre transform $H_0 f L$.

From now on, we will always assume all maps are assumed to be smooth, unless stated otherwise.

It turns out that the general Hamilton equation carries a least action principle on the phase space $T^*V$, which we now explain. The classical phase space corresponds to $T^*\mathbb{R}^n = \mathbb{R}^n \times (\mathbb{R}^n)^*$.

**Definition 1.3.18** (The action functional on phase space) Let $[t_0, t_1]$ be an interval and $\lambda : [t_0, t_1] \to T^*V$ be a curve. The action of $\lambda$ is defined to be

$$ \mathcal{A}_H(\lambda) = \int_{t_0}^{t_1} p \, dq - \int_{t_0}^{t_1} H(t, \lambda(t)) \, dt. \tag{1.3.25} $$

We call $\mathcal{A}_H$ the Hamiltonian action functional or just the (perturbed) action functional associated with the Hamiltonian function $H$. 

The following proposition shows that the Hamiltonian action functional specializes to the Lagrangian action functional via the Legendre transform. The proof is immediate and hence has been omitted.

**Proposition 1.3.19** Suppose that $H$ is the Legendre transform of $L$ and let $L$ be the Legendre transform map of $L$. Let $\gamma : [t_0, t_1] \to V$ be any path and let $\lambda = \tilde{\gamma}$ be its prolongation of $\gamma$. Then we have

$$A_H(\lambda) = \int_{t_0}^{t_1} L(t, \gamma(t), \dot{\gamma}(t)) dt.$$ 

A straightforward computation proves the following first-variation formula of $A_H$, given in (1.3.25).

**Exercise 1.3.20** Prove the following first-variation formula:

$$\delta A_H = p \cdot \delta x|_{t_0}^{t_1} - \int_{t_0}^{t_1} \left( \dot{p} \cdot \delta x - \dot{x} \cdot \delta p + \frac{\partial H}{\partial x} \delta x + \frac{\partial H}{\partial p} \delta p \right) dt. \quad (1.3.26)$$

Here we follow the physicist’s notations by denoting the variation vector of $\lambda$ by $\xi = (\delta x, \delta p)$ and $\delta A_H := dA_H(\xi)$. This formula gives rise to the following least action principle on the phase space.

**Theorem 1.3.21** Consider either of the following two cases.

1. **(Two-point boundary condition)** Let $H : [t_0, t_1] \times T^*V \to \mathbb{R}$ be any Hamiltonian. Put $x(t_0) = x_0$, $x(t_1) = x_1$ for given fixed points $x_0, x_1 \in V$.

2. **(Periodic boundary condition)** Assume that $H : \mathbb{R} \times T^*V \to \mathbb{R}$ is time-periodic, i.e., satisfies

$$H(t_0, x, p) \equiv H(t_1, x, p)$$

for all $(x, p) \in T^*V$. Put $x(t_0) = x(t_1)$ and $p(t_0) = p(t_1)$.

Then a path $\lambda : [t_0, t_1] \to T^*V$ satisfies Hamilton’s equation if and only if $\lambda$ is an extremal of $\mathcal{A}_H$, i.e., satisfies $\delta \mathcal{A}_H(\lambda) = 0$.

**Proof** The proof is an immediate consequence of the first-variation formula (1.3.26). \qed

*From now on, unless stated otherwise, we will always consider the unit interval $[0, 1]$ as the domain of a path $\lambda$. It is also customary to require $H$ that is one-periodic when we consider the periodic boundary condition.*
1.4 Classical Poisson brackets

Recall that Hamilton’s classical equation associated with a function $H$, which may or may not be time-dependent, is the first-order ordinary differential equation associated with the vector fields

$$\frac{\partial H}{\partial \dot{q}_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j}$$

on the phase space $\mathbb{R}^n \times (\mathbb{R}^n)^* \cong \mathbb{R}^{2n}$. We denote this vector field by $X_H$ and call it the Hamiltonian vector field associated with the function $H$.

The historical development of Hamiltonian mechanics and classical physicists and mathematicians’ attempts to find explicit solutions for the various mechanical systems reveal that the following algebraic structure on the space of differentiable functions is useful. This turns out to be the fundamental background geometric structure governing the Hamiltonian mechanics.

**Definition 1.4.1 (Classical Poisson bracket)** Let $F$ and $H$ be two functions on $\mathbb{R}^n \times (\mathbb{R}^n)^*$. The Poisson bracket, denoted by $\{F, H\}$, between $F$ and $H$ is defined by the formula

$$\{F, H\} = df(X_H) = \sum_j \left( \frac{\partial F}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q_j} \frac{\partial F}{\partial p_j} \right). \tag{1.4.27}$$

From the formula, it is manifest that the Poisson bracket defines a map

$$\{\cdot, \cdot\} : C^\infty(\mathbb{R}^{2n}) \times C^\infty(\mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^{2n}).$$

The following proposition summarizes the properties of this bracket.

**Proposition 1.4.2** The bracket satisfies the following properties:

1. **(Skew symmetry)** $\{F, H\} = -\{H, F\}$
2. **(Bilinearity)** $\{F + G, H\} = \{F, H\} + \{G, H\}$
3. **(Leibniz rule)** $\{FG, H\} = F\{G, H\} + \{F, H\}G$
4. **(Jacobi identity)** $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$
5. **(Nondegeneracy)** $\{F, G\} = 0$ for all $G$ if and only if $F \equiv C$ for some constant function $C \in \mathbb{R}$.

The properties (1)–(3) are easy to see from the formula (1.4.27). For the proofs of (4) and (5), we need some preparations.
Let \( X(\mathbb{R}^{2n}) \) be the set of vector fields on \( \mathbb{R}^{2n} \) and \( \text{ham}(\mathbb{R}^{2n}) \) the subset of Hamiltonian vector fields, i.e.,

\[
\text{ham}(\mathbb{R}^{2n}) = \{ X \mid X = X_H, H \in C^\infty(\mathbb{R}^{2n}) \}.
\]

We denote by \( \text{ev}_{(q,p)} : X(\mathbb{R}^{2n}) \to T_{(q,p)}\mathbb{R}^{2n} \cong \mathbb{R}^{2n} \) the evaluation map defined by \( \text{ev}_{(q,p)}(X) = X(q, p) \). The following lemma shows the ampleness of \( \text{ham}(\mathbb{R}^{2n}) \) on the tangent space.

**Lemma 1.4.3** At each point \((q, p) \in \mathbb{R}^{2n}\), we have

\[
\text{ev}_{(q,p)}(\text{ham}(\mathbb{R}^{2n})) = T_{(q,p)}\mathbb{R}^{2n} \cong \mathbb{R}^{2n}.
\]

**Proof** This is an immediate consequence of the following basic formulae:

\[
X_{q^i} = -\frac{\partial}{\partial p_j}, \quad X_{p_j} = \frac{\partial}{\partial q^j}.
\] (1.4.28)

\[\square\]

**Proof of nondegeneracy** (5) From the formula (1.4.27), we have \( \{ C, H \} = 0 \) for all \( H \) if \( C \) is a constant function. For the converse, we consider \( H = p_j \) or \( H = q^j \). We then have

\[
\{ F, p_j \} = dF(X_{p_j}) = dF\left( \frac{\partial}{\partial q^j} \right) = \frac{\partial F}{\partial q^j}
\]

and

\[
\{ F, q^j \} = dF(X_{q^j}) = -dF\left( \frac{\partial}{\partial p_j} \right) = -\frac{\partial F}{\partial p_j}.
\]

Therefore, if \( \{ F, H \} = 0 \) for all \( H \), then in particular we will have

\[
\frac{\partial F}{\partial q^j} = \frac{\partial F}{\partial p_j} = 0.
\]

This proves that \( F \) must be constant.

\[\square\]

Finally one can directly prove the Jacobi identity from the definition (1.4.27). Instead of carrying out this direct calculation, we will postpone its proof to the later chapters in which we will prove the Jacobi identity in the general context.

The Poisson bracket plays an important role in finding conserved quantities in the mechanical systems in the Hamiltonian mechanics, which in turn plays a fundamental role in the study of completely integrable systems. It also
plays a fundamental role in quantizing the classical mechanics into the quantum mechanics. We refer the reader to (Ar89) for a general discussion on the completely integrable systems and solving them by integration by quadratures, and (Di58) for the quantization process via the Poisson brackets. It turns out that the Poisson bracket is the geometric structure hidden in $\mathbb{R}^{2n}$ that governs the Hamiltonian mechanics, which we will generalize into the symplectic manifolds in the following chapters.
In this chapter, we will provide a globalization of the discussions of the previous sections in two stages.

- Replace $V$ by a manifold $N$ and generalize the Lagrangian mechanics.
- Replace $V \times V^*$ by an arbitrary even-dimensional manifold $M$ and generalize the Hamiltonian mechanics.

The first-stage generalization goes through without introducing any additional structure, but the second requires an additional geometric structure, the symplectic structure.

### 2.1 The cotangent bundle

We start with the observation that the Hamiltonian action functional

$$\mathcal{A}_H(\lambda) = \int_A p \, dq - \int_0^1 H \, dt$$

can be generalized to the cotangent bundle $T^*N$ of an arbitrary manifold $N$. Noting that the Legendre transformation still can be defined on the tangent bundle $TN$ for fiberwise convex functions, our discussion of the Hamiltonian action functional includes that of the Lagrangian action functional on $TN$. Therefore we will focus our discussion on the cotangent bundle. We refer the reader to Fathi’s book (Fa05) for a nice mathematical exposition on the Lagrangian dynamical system on general manifolds $N$.

We denote by $\pi : T^*N \to N$ the canonical projection. Let $(q^1, \ldots, q^n)$ be any local coordinate system on $U \subset N$, and its canonical coordinates
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\[(q^1 \circ \pi, \ldots, q^n \circ \pi, p_1, \ldots, p_n)\]
defined on \(T^* U = T^* N|_U \subset T^* N\). Here we abuse the notation just by denoting \(q^j = q^j \circ \pi\), whenever there is no danger of confusion. Recall that the fiber coordinates \((p_1, \ldots, p_n)\) are the linear coordinates with respect to the local frame \(\{dq^1, \ldots, dq^n\}\) of \(T^* N\) over the open set \(U\). We consider the one-form

\[\sum_j p_j d(q^j \circ \pi)\]  

(2.1.1)
defined on \(T^* U\). We leave the proof of the following as an exercise.

**Lemma 2.1.1** The locally defined one-form (2.1.1) is globally defined. More precisely, the following holds. For another system of canonical coordinates

\[(Q^1 \circ \pi, \ldots, Q^n \circ \pi, P^1, \ldots, P^n),\]
where \((Q^1, \ldots, Q^n)\) is a local coordinate system on \(V \subset N\), we have

\[\sum_j p_j d(q^j \circ \pi) = \sum_j P_j d(Q^j \circ \pi)\]
on the overlap \(U \cap V\).

**Exercise 2.1.2** Prove Lemma 2.1.1.

This suggests that there should be an invariant description of the globally defined one-form.

**Definition 2.1.3** (Liouville one-form) Define a one-form, denoted by \(\theta\), on \(T^* N\) by the formula

\[\theta_\alpha(\xi) = \alpha(d\pi(\xi)),\]
where \(\alpha \in T^* N\) and \(\xi \in T_\alpha(T^* N)\). This one-form is called the Liouville one-form.

It immediately follows that, in canonical coordinates \((q^1, \ldots, q^n, p_1, \ldots, p_n)\), \(\theta\) is precisely written as (2.1.1). The following properties characterize the Liouville one-form \(\theta\) whose proof we leave as an exercise.
Proposition 2.1.4  The Liouville form $\theta$ satisfies the following properties.

1. Let $\alpha$ be any one-form on $N$. Then
   \[ \tilde{\alpha}^* \theta = \alpha \]  
   on $N$. Here, on the left-hand side, we denote by $\tilde{\alpha}$ the map $N \to T^*N$ induced by the section $\alpha$ of the bundle $\pi : T^*N \to N$.
2. $\theta(\xi) = 0$ for any vertical tangent vector $\xi$, i.e., those $\xi$ satisfying $d\pi(\xi) = 0$.

Conversely, any one-form satisfying the above two properties coincides with $\theta$.

Proof  It is easy to check the two properties by definition of $\theta$. We will just prove the converse.

Consider any one-form $\theta'$ on $T^*N$ that satisfies the two properties. Let $x \in T^*N$ be an arbitrary point and let $q = \pi(x)$. Pick any one-form $\alpha$ on $N$ with $\tilde{\alpha}(q) = x$. Then
\[ T_x(T^*N) = VT_x(T^*N) \oplus d\tilde{\alpha}(T_qN), \]
where $VT_x(T^*N)$ is the vertical tangent space at $x \in T^*N$. The property (2) implies $\theta'(x)|_{VT_x(T^*N)} \equiv \theta(x)|_{VT_x(T^*N)}$. Therefore it suffices to prove
\[ \theta'(x)(d\tilde{\alpha}(v)) = \theta(x)(d\tilde{\alpha}(v)) \]
for all $v \in T_qN$. But this equality is equivalent to $\tilde{\alpha}^* \theta'(q)(v) = \tilde{\alpha}^* \theta(q)(v)$. But the latter obviously holds because property (1) implies $\tilde{\alpha}^* \theta' = \alpha = \tilde{\alpha}^* \theta$, which finishes the proof.

We would like to note that the statements (1) and (2) together imply
\[ \theta \equiv 0 \]  
(2.1.3)
on the zero section.

It turns out that a more fundamental geometric structure that exists in the more general context is the one induced by the differential of $\theta$, which is called a symplectic form.

Definition 2.1.5 (Canonical symplectic form)  Let $N$ be an arbitrary manifold. The canonical symplectic form, denoted by $\omega_0$, of $T^*N$ is defined by
\[ \omega_0 = -d\theta. \]

The choice of the negative sign in front of $d\theta$ is to make the coordinate expression of $\omega_0$ coincide with
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\[ \sum_j dq^j \wedge dp_j, \]

not with \( \sum_j dp_j \wedge dq^j \) as some authors do. By definition, \( \omega_0 \) is closed (indeed exact) and defines a skew-symmetric bilinear two-form at each tangent space \( T_\alpha(T^*N) \). The following nondegeneracy is less trivial to see.

**Lemma 2.1.6** The two-form \( \omega_0 \) is nondegenerate in that, at any point \( \alpha \in T^*N, \xi = 0 \) if and only if \( \xi \lbrack \omega_0 = 0 \) for \( \xi \in T_\alpha(T^*N) \).

**Proof** Since \( \omega_0 \) is a tensor, we first express it in terms of the canonical coordinates \((q^1, \ldots, q^n, p_1, \ldots, p_n)\). It follows from Exercise 2.1.2 that

\[ \omega_0 = \sum_j dq^j \wedge dp_j, \]

from which the nondegeneracy is manifest. \( \square \)

### 2.2 Symplectic forms and Darboux’ theorem

We are now ready to introduce the general definition of symplectic forms and symplectic manifolds. Let \( M \) be a smooth manifold.

#### 2.2.1 Definition of symplectic form

**Definition 2.2.1** A *symplectic form* on \( M \), denoted by \( \omega \), is a nondegenerate, closed two-form defined on \( M \). The pair \((M, \omega)\) is called a *symplectic manifold*.

**Definition 2.2.2** Let \((M_1, \omega_1)\) and \((M_2, \omega_2)\) be symplectic manifolds. A smooth (or more generally \( C^1 \)) map \( \psi : M_1 \to M_2 \) is called *symplectic* if \( \psi^* \omega_2 = \omega_1 \). When \( \psi \) is a diffeomorphism, we call it a *symplectic diffeomorphism* or simply a *symplectomorphism*.

Note that any symplectic map must be automatically *immersed*. Just as differential topology concerns studying the invariants of manifolds up to diffeomorphisms, *symplectic topology* concerns the invariants of symplectic manifolds up to the symplectic diffeomorphisms. One of the first things known about the symplectic geometry was the fact that locally the dimension is the only invariant of the symplectic manifolds, just like in the topology. This is known as *Darboux’ theorem*. We will prove the Darboux theorem later in this section.
Before proceeding further, we provide some examples

**Example 2.2.3**

1. We have already shown that the cotangent bundle $T^*N$ of any smooth manifold $N$ carries the canonical symplectic form $\omega_0$. This includes the classical phase space $\mathbb{R}^n \times (\mathbb{R}^n)^*$ as a special case.

2. Consider any orientable two-dimensional surface $\Sigma$ and an area form $\omega$, i.e., a nowhere-vanishing two-form. Recall that the existence of such a two-form is guaranteed by the orientability assumption. Then $\omega$ is a symplectic form.

3. The complex projective space, or more generally any Kähler manifold in the complex geometry, carries a canonical symplectic form, the given Kähler form.

The following theorem by Gompf in particular shows that we have a fairly large class of four manifolds that allow a symplectic structure.

**Theorem 2.2.4** (Gompf (Gom95)) *Any finitely presented group can be realized by the fundamental group of a symplectic four manifolds.*

Now we analyze the consequences of each condition imposed on the definition of symplectic forms.

We start with the consequence of $\omega$ being a differential two-form, i.e., a skew-symmetric covariant two-tensor. We do not assume that $\omega$ is either nondegenerate or closed at this time. Such a two-form induces a bundle map $\widetilde{\omega} : TM \to T^*M; X \mapsto \omega(X, \cdot)$.

This is *skew-symmetric*: We recall that on a finite-dimensional vector space $V$ we have a canonical isomorphism $(V^*)^* = V$. Therefore the adjoint $A^* \in \text{Hom}((V^*)^*, V^*) \equiv \text{Hom}(V, V^*)$ can be compared with $A \in \text{Hom}(V, V^*)$. This gives rise to the following general definition.

**Definition 2.2.5** A linear map $A : V \to V^*$ is called skew-symmetric if $A = -A^*$.

Nondegeneracy of $\omega$ then implies that $\widetilde{\omega} : TM \to T^*M$ is a skew-symmetric isomorphism. Recalling that any skew-symmetric quadratic form has even rank, nondegeneracy of $\omega$ forces $M$ to be of even dimension. In other words, we have the following.
Corollary 2.2.6 If $M$ carries a symplectic form, then $M$ must have even dimension and carries a canonical volume form $(1/n!)\omega^n$. In particular, $(M, \omega)$ is orientable and canonically oriented.

We call the above volume form the \textit{Liouville volume form} and the associated measure the \textit{Liouville measure}. The orientation induced by the Liouville volume form on $\mathbb{R}^{2n}$ coincides with the complex orientation of $\mathbb{C}^n$ if we identify $\mathbb{R}^{2n} \cong \mathbb{R}^n \times (\mathbb{R}^n)^*$ by the map

$$z_j = q^j + \sqrt{-1}p_j, \quad j = 1, \ldots, n.$$ 

Nondegeneracy also allows us to give the following definition

Definition 2.2.7 Let $\omega$ be any nondegenerate two-form on $M$, not necessarily closed. Consider $h \in C^\infty(M)$.

(1) The \textit{quasi-Hamiltonian vector field}, associated with $h$, denoted by $X_h$, is the vector field defined by

$$X_h = \tilde{\omega}^{-1}(dh). \quad (2.2.4)$$ 

(2) The \textit{quasi-Poisson bracket}, denoted by $\{f, h\}$, is defined by

$$\{f, h\} = \omega(X_f, X_h). \quad (2.2.5)$$ 

We first mention that we can equivalently write

$$\{f, h\} = df(X_h) = X_h[f]. \quad (2.2.6)$$

It immediately follows from the definition that the quasi-Poisson bracket associated with any nondegenerate two-form satisfies skew-symmetry, bilinearity and the Leibnitz rule. The remaining question is under what condition on $\omega$ does the associated quasi-Poisson bracket satisfy the Jacobi identity? The following is a consequence of nondegeneracy.

Exercise 2.2.8 Prove that the set of quasi-Hamiltonian vector fields is ample in that the following holds. Let $x \in M$ be any given point. Then we have

$$\{X(x) \mid X = X_h, \ h \in C^\infty(M)\} = T_xM.$$

Here is the first consequence of the closedness, which is of dynamical nature.
Theorem 2.2.9  Let $\omega$ be a nondegenerate two-form and $\{\cdot, \cdot\}$ be its associated quasi-Poisson bracket. Then $\omega$ is closed if and only if the quasi-Poisson bracket satisfies the Jacobi identity.

Proof  From the definition of the exterior derivative, we have

$$d\omega(X, Y, Z) = X[\omega(Y, Z)] - Y[\omega(X, Z)] + Z[\omega(X, Y)] - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X).$$  \hspace{1cm} (2.2.7)

We derive the following general identity for any nondegenerate closed two-form.

Proposition 2.2.10  Let $\omega$ be as in the theorem. Then

$$d\omega(X_f, X_g, X_h) = -\{(g, h), f\} + \{(h, f), g\} + \{(f, g), h\}. \hspace{1cm} (2.2.8)$$

Proof  Substituting $X = X_f$, $Y = X_g$, $Z = X_h$, we derive

$$d\omega(X_f, X_g, X_h) = X_f[\omega(X_g, X_h)] - X_g[\omega(X_f, X_h)] + X_h[\omega(X_f, X_g)] - \omega([X_f, X_g], X_h) + \omega([X_f, X_h], X_g) - \omega([X_g, X_h], X_f).$$

The first line becomes

$$\{(g, h), f\} + \{(h, f), g\} + \{(f, g), h\}$$

from the definition of the bracket. On the other hand, we compute

$$-\omega([X_f, X_g], X_h) = dh([X_f, X_g]) = [X_f, X_g](h) = (L_{X_f}L_{X_g} - L_{X_g}L_{X_f})(h) = \{(h, g), f\} - \{(h, f), g\}$$

and similarly

$$\omega([X_f, X_h], X_g) = -\{(g, h), f\} + \{(g, f), h\},$$

$$-\omega([X_g, X_h], X_f) = \{(f, h), g\} - \{(f, g), h\}.$$

By adding them up, we have derived (2.2.8). \hfill \Box

From this proposition, it immediately follows that closedness of $\omega$ implies the Jacobi identity.

The converse also follows from (2.2.8) together with the ampleness of the set of quasi-Hamiltonian vector fields (Exercise 2.2.8) whose detail is in order. We first get

$$d\omega(X_f, X_g, X_h) = -\{(f, g), h\} - \{(h, f), g\} - \{(g, h), f\} = 0$$
for all \( f, g, h \) from (2.2.8). At any point \( x \in M \), we evaluate \( d\omega(u, v, w) \) against the three quasi-Hamiltonian vector fields \( X_f, X_g, X_h \) satisfying \( X_f(x) = u, X_g(x) = v, X_h(x) = w \). This proves that the Jacobi identity implies the closedness of \( \omega \) and finishes the proof of the theorem.

We now summarize the basic properties of the Poisson bracket associated with the symplectic structure \( \omega \). We first provide the general definition of the Poisson structure on a manifold.

**Definition 2.2.11 (Poisson bracket)** Let \( M \) be a manifold. A Poisson structure on \( M \) is a bilinear map, called a Poisson bracket,

\[
\{ \cdot, \cdot \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M),
\]

that satisfies the following properties:

1. **(Skew symmetry)** \( \{ f, h \} = -\{ h, f \} \)
2. **(Bilinearity)** \( \{ f + g, h \} = \{ f, h \} + \{ g, h \} \)
3. **(Leibniz rule)** \( \{ fg, h \} = f \{ g, h \} + \{ f, h \} g \)
4. **(Jacobi identity)** \[ \{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = 0. \]

We call a manifold \( M \) equipped with a Poisson structure a Poisson manifold.

We refer the reader to (Wn83) for an introduction to the general theory of the Poisson manifold, and to (Kon03) for the fundamental formality theorem on the Poisson structure and its deformation quantization.

With this general definition, we can state the following theorem.

**Theorem 2.2.12** Let \((M, \omega)\) be a symplectic manifold. Define a bilinear map

\[
\{ \cdot, \cdot \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M),
\]

by the formula

\[
\{ f, h \} = \omega(X_f, X_h) = df(X_h).
\]

(2.2.9)

Then this defines a Poisson bracket in the sense of Definition 2.2.11. In addition, the Poisson structure associated with a symplectic structure satisfies the following additional property:

- **(Nondegeneracy)** \( \{ f, g \} = 0 \) for all \( g \) if and only if \( f \equiv C \) for some constant function \( C \in \mathbb{R} \).

**Proof** From the definition (2.2.9), skew symmetry, bilinearity and the Leibniz rule are evident. On the other hand, we have proven the Jacobi identity in Theorem 2.4.8. It remains to prove nondegeneracy. The proof is similar to that...
of the classical case. It relies on the ampleness of the set of quasi-Hamiltonian vector fields, and is omitted.

The equality \( \{ f, h \} = df(X_h) = X_h[f] \) shows that, as a derivation acting on \( C^\infty(M) \), we have the equality

\[
\{ \cdot, h \} = X_h.
\] (2.2.10)

Here we would like to emphasize that in our conventions the Hamiltonian vector field \( X_h \) corresponds to the bracket taken with \( h \) in the second spot, not in the first spot.

Now we can finish the proof of the Jacobi identity of the classical Poisson bracket on the phase space, as a special case of Theorem 2.2.9 which we have postponed till this section.

**Corollary 2.2.13** The classical Poisson bracket given in Definition 1.4.1 satisfies the Jacobi identity.

**Proof** We have only to note that the classical Hamiltonian vector field given by the formula

\[
\sum_{i=1}^{n} \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i}
\]

is nothing but the Hamiltonian vector field of \( f \) associated with the canonical symplectic form

\[
\omega_0 = \sum_j dq^j \wedge dp_j
\]
on \( \mathbb{R}^{2n} \), which is closed. Then Theorem 2.2.9 will finish the proof. □

Another immediate consequence of the closedness of \( \omega \), this time of topological nature, is the following

**Proposition 2.2.14** Let \((M, \omega)\) be a closed symplectic manifold. Then the de Rham cohomology class \([\omega]\) ≠ 0 in \( H^2(M, \mathbb{R}) \). In particular, we have \( H^2(M, \mathbb{R}) \neq 0 \).

**Proof** We note that the top power \( \omega^n \) is a nowhere-vanishing 2n-form and so defines a volume form. Since \( M \) is closed and \( \omega^n \) is a nowhere-vanishing top form, we know that \( \int_M \omega^n > 0 \) and so \([\omega^n]\) ≠ 0 in \( H^{2n}(M, \omega) \). Using the fact that \( H^*(M, \mathbb{R}) \) forms a ring under the wedge product, we have

\[
[\omega]^n = [\omega^n] \neq 0
\]
and hence \([\omega]\) ≠ 0. This finishes the proof. □
This proposition in particular implies that only the two-sphere $S^2$ can be symplectic among the set of $n$-spheres. Since $S^2$ is orientable, it carries an area form that is nondegenerate and closed, i.e., a symplectic form.

Another fundamental consequence of closedness is the Darboux theorem whose proof will occupy the rest of this section. We start with some basic linear algebra.

### 2.2.2 Symplectic linear group $Sp(2n)$

Consider any symplectic vector space $(S, \Omega)$ equipped with a nondegenerate skew-symmetric bilinear form.

A linear map $A : S \to S$ is called symplectic if it satisfies $A^\ast \Omega = \Omega$. We note that any such map is automatically invertible.

**Definition 2.2.15** We denote by $Sp(S, \Omega)$ the set of symplectic automorphisms of $(S, \Omega)$.

It follows that $Sp(S, \Omega)$ is a subgroup of $GL(S)$, the group of invertible automorphisms on $S$. An easy consequence of linear algebra is the following.

**Lemma 2.2.16** There exists a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of $S$ such that

$$\Omega(e_i, e_j) = 0 = \Omega(f_i, f_j), \quad \Omega(e_i, f_j) = \delta_{ij}. \quad (2.2.11)$$

We call any such basis a symplectic basis or a Darboux basis.

This lemma says that any $(S, \Omega)$ is isomorphic to the canonical symplectic vector space

$$S_V := V \oplus V^*$$

with the canonical symplectic inner product $\Omega_V$ defined by

$$\Omega_V((v_1, \alpha_1), (v_2, \alpha_2)) = \alpha_2(v_1) - \alpha_1(v_2). \quad (2.2.12)$$

For $(S_V, \Omega_V)$, any choice of a basis $\{e_1, \ldots, e_n\}$ of $V$ and its dual basis $\{f_1, \ldots, f_n\}$ gives rise to a symplectic basis on $S_V$. This choice then gives an isomorphism

$$(V \oplus V^*, \Omega_V) \cong \left( \mathbb{R}^{2n}, \sum_{i=1}^{2n} dx_i \wedge dy_i \right).$$

The $Sp(2n; \mathbb{R})$ is the automorphism group of $(\mathbb{R}^{2n}, \omega_0)$. 

Recall that \( \mathbb{R}^{2n} \) has two other fundamental geometric structures, the Euclidean inner product which we denote by \((\cdot, \cdot)\) and the complex multiplication \( i : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) which identifies \( \mathbb{R}^{2n} \) with \( \mathbb{C}^n \) by the coordinate relation

\[
z_j = x_j + iy_j.
\]

Then we have the following fundamental identities between their automorphisms:

\[
O(2n; \mathbb{R}) \cap S p(2n; \mathbb{R}) \cong O(2n; \mathbb{R}) \cap GL(n; \mathbb{C}) \cong GL(n; \mathbb{C}) \cap S p(2n; \mathbb{R}) \cong U(n).
\]

We now make these isomorphisms more precise.

We recall the embedding \( GL(n; \mathbb{C}) \hookrightarrow GL(2n; \mathbb{R}) \) defined by the realization map

\[
X + iY \mapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}.
\]

More precisely, for a given \( \mathbb{C}\)-linear map \( A \in GL(n; \mathbb{C}) \), its realization \( A_\mathbb{R} \) is the \( \mathbb{R}\)-linear map that commutes the diagram

\[
\begin{array}{ccc}
\mathbb{R}^n \otimes \mathbb{C} & \longrightarrow & \mathbb{R}^{2n} \\
\downarrow A & & \downarrow A_\mathbb{R} \\
\mathbb{R}^n \otimes \mathbb{C} & \longrightarrow & \mathbb{R}^{2n}
\end{array}
\]

Denote the image of a subset \( K \subset GL(n; \mathbb{C}) \) in \( GL(2n; \mathbb{R}) \) by \( K_\mathbb{R} \) and call it the realization of \( K \).

The isomorphism \( O(2n; \mathbb{R}) \cap S p(2n; \mathbb{R}) \cong U(n) \) can be explicitly seen in the following lemma.

**Lemma 2.2.17** A matrix \( P \in GL(2n; \mathbb{R}) \) lies in \( O(2n; \mathbb{R}) \cap S p(2n; \mathbb{R}) \) if and only if

\[
P = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}
\]

with \( X'Y - Y'X = 0 \) and \( X'X + Y'Y = I \) or equivalently \( P = A_\mathbb{R} \) with \( A = X + iY \in U(n) \). In other words, we have

\[
O(2n; \mathbb{R}) \cap S p(2n; \mathbb{R}) = U(n)_\mathbb{R}.
\]

With the identification of \( GL(n; \mathbb{C}) \) with its realization \( GL(2n; \mathbb{R}) \), we have the identity \( GL(n; \mathbb{C})_\mathbb{R} \cap S p(2n; \mathbb{R}) = U(n)_\mathbb{R} \).
Finally we recall that $\mathcal{U}(n)$ is defined to be the automorphism group of the Hermitian inner product $\langle \cdot , \cdot \rangle$ defined by

$$
\langle v, w \rangle = \sum_{i=1}^{n} v_i \bar{w}_i \tag{2.2.14}
$$

for $v = (v_1, \ldots, v_n) \in \mathbb{C}^n$ and similarly for $w$. Note that this Hermitian inner product is complex linear for the first argument and anti-complex linear for the second. We denote by $z_\mathbb{R}$ the image of the isomorphism $z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$ and call $z_\mathbb{R}$ the \textit{realization} of the complex vector $z$.

**Lemma 2.2.18** Let $v, w \in \mathbb{C}^n$ and let $v_\mathbb{R}, w_\mathbb{R}$ be their respective realizations. Then we have

$$
\langle v, w \rangle = (v_\mathbb{R}, w_\mathbb{R}) - i\omega_0(v_\mathbb{R}, w_\mathbb{R}).
$$

In particular, we have $O(2n; \mathbb{R}) \cap S p(2n; \mathbb{R}) = \mathcal{U}(n)_\mathbb{R}$.

Another important structure theorem of $S p(2n; \mathbb{R})$ is that $\mathcal{U}(n)_\mathbb{R}$ is the maximal compact subgroup and hence $S p(2n; \mathbb{R})/(\mathcal{U}(n)_\mathbb{R})$ is contractible by the general theory of Lie groups. (See (He78), for example. We refer readers to (CZ84, SZ92) for a construction of an explicit deformation retraction of $S p(2n; \mathbb{R})$ to $(\mathcal{U}(n)_\mathbb{R})$. This implies that the homotopy type of $S p(2n; \mathbb{R})$ is the same as $\mathcal{U}(n)$. Fixing any deformation retraction of $S p(2n; \mathbb{R})$ to $\mathcal{U}(n)_\mathbb{R}$, we can extend the determinant mapping $\det : \mathcal{U}(n) \to S^1$ to

$$
\tilde{\det} : S p(2n; \mathbb{R}) \to S^1,
$$

which induces an isomorphism in $\pi_1$. Obviously this map is continuous but not a homomorphism, unlike $\det U(n) \to S^1$. (See (CZ84, SZ92) for relevant discussion on this issue.) This is because the deformation retraction $S p(2n; \mathbb{R}) \to \mathcal{U}(n)_\mathbb{R}$ does not preserve the group structure. (See (SZ92).)

**Exercise 2.2.19** Prove that there is no deformation retraction from $S p(2n; \mathbb{R})$ to $\mathcal{U}(n)$ that preserves the group structure.

**Definition 2.2.20** Let $\gamma$ be a loop in $S p(2n; \mathbb{R})$. The index $\mu(\gamma)$ is defined to be the degree of the map $\tilde{\det} \circ \gamma : S^1 \to S^1$. 
2.2 Symplectic forms and Darboux’ theorem

2.2.3 Proof of Darboux’ theorem

We now state the fundamental Darboux theorem of symplectic manifolds.

**Theorem 2.2.21 (Darboux’ theorem)** Let \((M, \omega)\) be an arbitrary symplectic manifold of dimension \(2n\). Then at any point \(x_0 \in M\) there exists a coordinate chart \(\varphi : U \to \mathbb{R}^{2n}\) with \(x_0 \in U\) such that \(\omega = \varphi^* \omega_0\), i.e., we have

\[
\omega = \sum_{i=1}^{n} dx_i \wedge dy_i
\]

for \(\varphi = (x_1, \ldots, x_n, y_1, \ldots, y_n)\). We call any such coordinates the Darboux coordinates.

**Proof** The scheme of the proof uses one of the fundamental techniques in symplectic geometry, which is known as Moser’s deformation method (Mo65).

As a first step, we choose a diffeomorphism \(\psi : U' \subset M \to V' \subset T^* \mathbb{R}^n \cong \mathbb{R}^{2n}\) such that

\[
\omega|_{x_0} = \psi^* \omega_0|_{x_0},
\]

at the given point \(x_0 \in U'\). This can be achieved by using the simple fact that \(GL(2n)\) acts transitively on the set of skew-symmetric bilinear forms on \(\mathbb{R}^{2n}\) under the congruence action.

To achieve (2.2.15), we first choose any local chart \(\tilde{\psi} : \tilde{U} \to \tilde{V}\) with \(x_0 \in \tilde{U} \subset U'\). Then we have two nondegenerate quadratic forms on \(T_{\tilde{\psi}(x_0)} \mathbb{R}^{2n} \cong \mathbb{R}^{2n}\), one \(\omega_0\) and the other \(\omega_1 := \tilde{\psi}_* \omega\) on \(\mathbb{R}^{2n}\). Since any two nondegenerate skew-symmetric bilinear forms are congruent to each other, it follows that there is a matrix \(A \in GL(2n)\) such that

\[
A^* \omega_1 := \omega_1 (A\cdot, A\cdot) = \omega_0,
\]

Then we choose \(\psi = A^{-1} \circ \tilde{\psi}\), which will satisfy (2.2.15).

In the second step of the proof, we deform \(\psi\) to the required \(\varphi : U \to \mathbb{R}^{2n}\) by composing \(\psi\) with a diffeomorphism \(h : U \to U'\) such that

\[
h^*(\psi^* \omega_0) = \omega
\]

on \(U \subset U'\). Then \(\varphi = \psi \circ h\) will achieve our purpose.

For the latter purpose, we will prove a general deformation lemma due to Weinstein (Wn73). This is more than what we need to prove the Darboux theorem, for which \(N\) corresponds to \(N = \{x_0\}\), a point.

**Lemma 2.2.22 (Deformation lemma)** Let \(N \subset M\) be a compact submanifold, and let \(\omega\) and \(\omega'\) be two symplectic forms defined in a neighborhood of \(N\). If
\[ \omega \equiv \omega' \text{ on } TM|_N, \text{ then there exists a diffeomorphism } h : U \to U' \text{ with both } U, U' \text{ being a neighborhood of } N \text{ such that} \]

(1) \[ dh|_{TM|_N} \equiv id|_{TM|_N} \]

(2) \[ h^* \omega' = \omega \text{ on } U \supset N. \]

Moser’s deformation method Consider the one-parameter family \( \omega_t = (1 - t)\omega + t\omega', 0 \leq t \leq 1. \) We note that \( \omega_t|_{TM|_N} \equiv \omega|_{TM|_N} (= \omega'|_{TM|_N}) \) for all \( 0 \leq t \leq 1. \) In particular, \( \omega_t \) is nondegenerate in a small neighborhood of \( N, \) i.e., symplectic for all \( 0 \leq t \leq 1 \) there. We will seek a family of diffeomorphisms \( \{h_t\}_{0 \leq t \leq 1} \) defined near \( N \) so that

\[
\begin{cases}
    h_t^* \omega_t = \omega, \\
    dh_t|_{TM|_N} = id \text{ on } TM|_N.
\end{cases}
\]

(2.2.16)

The key point of Moser’s deformation method (Mo65) is that, instead of solving this problem, we transform the problem into that of the family of vector fields generating the isotopy \( h_t, \) i.e., \( \xi_t = dh_t/dt \circ h_t^{-1}. \) Of course, once we have obtained the family \( \xi_t, \) we can recover \( h_t \) by solving the initial value problem of the ODE

\[
\frac{dh_t}{dt} = \xi_t \circ h_t, \quad h_0 = id.
\]

To write down the condition for \( \xi_t \) to satisfy, we differentiate (2.2.16) in \( t: \)

\[
0 = \frac{d}{dt} h_t^* \omega_t = h_t^* \left( L_{\xi_t} \omega_t + \frac{d\omega_t}{dt} \right)
= h_t^* \left( d(\xi_t)^\omega_t + \frac{d\omega_t}{dt} \right).
\]

We remark that, since \( h_t|_N = id|_N \) and \([0, 1] \) is compact, \( h_t \) maps a fixed neighborhood \( U_1 \) of \( N \) into another bigger neighborhood \( U_2 \) of \( N \) for every \( t \in [0, 1]. \) Therefore we can get rid of \( h_t^* \) and obtain

\[
\begin{cases}
    d(\xi_t)^\omega_t + d\omega_t/dt = 0 \text{ on } U_1, \\
    \xi_t \equiv 0.
\end{cases}
\]

(2.2.17)

Now, by setting \( \beta_t = \xi_t|_\omega_t \) and noting that \(-d\omega_t/dt = \omega - \omega', \) we further transform (2.2.17) into

\[
\begin{cases}
    d\beta_t = \omega - \omega' \quad \text{in a neighborhood of } N, \\
    \beta_t \equiv 0 \quad \text{on } N.
\end{cases}
\]

(2.2.18)

Here the neighborhood itself on which the equation \( d\beta_t = \omega - \omega' \) is to be solved will be determined. As an easy application of the tubular neighborhood
theorem, we can find a deformation retraction of a tubular neighborhood $U_1 \supset N$ onto $N$

$$\psi_t : U_1 \to U, \ 0 \leq t \leq 1, \ \psi_t|_N = id_N$$

such that $\psi_t$, $t > 0$ are diffeomorphisms onto its image and $\psi_1 = id|_{U_1}$.

Now we denote $\Omega = \omega - \omega'$ and consider $\psi_t^* \Omega$. Then we have

$$\Omega = \psi_t^* \Omega = \psi_0^* \Omega + \int_0^1 \frac{d}{dt} \psi_t^* \Omega \, dt$$

$$= \psi_0^* \Omega + \int_0^1 \psi_t^*(\eta_t^* \Omega) \, dt$$

with

$$\eta_t = \begin{cases} \frac{d}{dt} \psi_t \circ \psi_t^{-1}, & t > 0, \\ 0, & t = 0. \end{cases}$$

By the hypothesis $\omega \equiv \omega'$ on $TM|_N$ and $\text{Im} \psi_0 \subset N$, we have $\psi_0^* \Omega = 0$. Therefore we have only to choose the form $\beta_t$ given by

$$\beta_t \equiv \int_0^t \psi_s^*(\eta_s^* \Omega) \, ds$$

independently of $t$. Notice that $\beta_t = 0$ on $N$ since $\psi_t|_N = id|_N$ and $\eta_t \equiv 0$ on $N$. This finishes the construction of $\beta_t$ satisfying (2.2.18), and hence the construction of the vector fields $\xi_t$ satisfying (2.2.17).

Finally, to obtain the isotopy $h_t$ in a neighborhood of $N$, we have to make sure that the domain $D \subset [0, 1] \times U_1$ of existence of the ODE $\dot{x} = \xi_t(x)$ defined on $[0, 1] \times U_1$ contains a subset of the form $[0, 1] \times U$ contained in $D$. Since $\xi_t \equiv 0$ on $N$ and $N$ is assumed to be compact, $D$ contains the subset $[0, 1] \times N$.

On the other hand, by the fundamental existence theorem of an ODE, $D$ is always an open subset of $[0, 1] \times U_1$. Since $[0, 1]$ is compact, it then follows that $D$ must contain a subset of the form $[0, 1] \times U$ with $N \subset U \subset U_1$. This proves that $h_t : U \to h_t(U)$ is well-defined for all $(t, x) \in [0, 1] \times U$ and satisfies the required property (2.2.16). \hfill $\square$

Once we have finished the proof of this general deformation lemma, we can finish the proof of Darboux’ theorem by applying $N = \{x\} \subset M$. \hfill $\square$

Now we go backwards and state Moser’s theorem (Mo65), which started this whole isotopy problem of symplectic forms. We first introduce the following general definitions.

**Definition 2.2.23** Assume that $M$ is closed and let $\omega_0$ and $\omega_1$ be two symplectic forms on $M$. 
(1) We say that $\omega_0$ and $\omega_1$ are *isotopic* if there exists a smooth family $\{\omega_t\}$ of symplectic forms such that their cohomology classes $[\omega_t]$ satisfy

$$[\omega_t] = [\omega_0] \quad \text{in } H^2(M, \mathbb{R}) \text{ for all } t \in [0, 1].$$

We call such a family $\{\omega_t\}$ an *isotopy* between $\omega_0$ and $\omega_1$.

(2) We say that $\omega_0$ and $\omega_1$ are *pseudo-isotopic* if there exists a smooth family of symplectic forms connecting them, not necessarily preserving their cohomology classes.

(3) We say that $\omega_0$ and $\omega_1$ are *deformation equivalent* if there exists a diffeomorphism $\varphi : M \rightarrow M$ such that $\omega_0$ and $\varphi^* \omega$ are pseudo-isotopic to each other.

**Exercise 2.2.24** (Moser) Assume $M$ is closed. For any two isotopic forms $\omega_0$ and $\omega_1$ and an isotopy $\{\omega_t\}$, there exists an isotopy $\varphi_t : M \rightarrow M$ of diffeomorphisms with $\varphi_0 = id$ such that $\varphi_t^* \omega_t = \omega_0$. In particular, the two forms $\omega_0$ and $\omega_1$ are diffeomorphic. (Hint. Use the Hodge decomposition theorem or see (Mo65).)

An immediate corollary of Darboux’ theorem is the following embedding theorem of small standard balls in $\mathbb{C}^n$ into $(M, \omega)$.

**Corollary 2.2.25** Let $(M, \omega)$ be an arbitrary symplectic manifold. There exists a symplectic embedding $h : B^{2n}(R) \subset \mathbb{C}^n \rightarrow (M, \omega)$ for a sufficiently small $R > 0$.

This leads us to the following notion of the Gromov radius of $(M, \omega)$.

**Definition 2.2.26** The *Gromov radius* is the supremum of $R$ for which there exists a symplectic embedding $h : B^{2n}(R) \subset \mathbb{C}^n \rightarrow (M, \omega)$. We denote it by $\hat{r}(M, \omega)$.

In this regard, Gromov’s theorem (Gr85) is the fundamental theorem which manifests an essential difference between symplectic maps and volume preserving maps.

**Theorem 2.2.27** (Gromov’s non-squeezing theorem) Consider the cylinder

$$Z_n(r) = D^2(r) \times \mathbb{C}^{n-1} \subset \mathbb{C}^n$$

with the standard symplectic form $\omega_0$. Then $\hat{r}(Z_n(r), \omega_0) = r$. 
2.3 The Hamiltonian diffeomorphism group

We refer readers to Section 11.1 for its complete proof, following Gromov’s original geometric proof via the method of pseudoholomorphic curves in (Gr85). We refer the reader to (EkH89) for a dynamical proof via the study of Hamiltonian periodic orbits. At this stage, we would like to emphasize that it would be most desirable to develop a method of incorporating both aspects of symplectic geometry. One of the main goals of this book is to convey this mathematical perspective via the chain-level Floer theory.

2.3 The Hamiltonian diffeomorphism group

Owing to the closedness of $\omega$ and Cartan’s ‘magic’ formula

$$L_X\alpha = d(X\lrcorner \alpha) + X\lrcorner d\alpha$$

we have $L_X\omega = d(X\lrcorner \omega)$ for general vector fields. In particular,

$$L_X f \omega = 0 \quad (2.3.19)$$

for any smooth function $f$ on $M$.

**Definition 2.3.1** A diffeomorphism $\phi : M \to M$ is called symplectic if $\phi^* \omega = \omega$. We denote by $\text{Symp}(M, \omega)$ the set of symplectic diffeomorphisms. We call a vector field $X$ symplectic if $L_X\omega = 0$, or, equivalently, if the one-form $X\lrcorner \omega$ is closed.

It is obvious that $\text{Symp}(M, \omega)$ forms a subgroup of $\text{Diff}(M)$. In fact, by definition, any symplectic diffeomorphism preserves the Liouville measure $\mu_\omega$ induced by the volume form $\omega^n/\omega!$.

**Lemma 2.3.2** (Liouville’s lemma) For any symplectic diffeomorphism $\phi$, it preserves Liouville’s measure, i.e., $\mu_\omega(U) = \mu_\omega(\phi(U))$ for any subset $U \subset M$. In particular, for any symplectic vector field $X$

$$L_X \omega^n = 0.$$

2.3.1 Definition of Ham($M, \omega$)

The definition of the group $\text{Symp}(M, \omega)$ is natural in that it is the structure group of the symplectic form $\omega$. We denote by $\text{Symp}_0(M, \omega)$ the identity component of $\text{Symp}(M, \omega)$. However, there is another more mysterious subgroup of $\text{Symp}_0(M, \omega)$, called the Hamiltonian diffeomorphism group.

We start by giving the definition of Hamiltonian vector fields.
Definition 2.3.3 Let \((M, \omega)\) be a symplectic manifold.

1. We call a vector field \(X\) Hamiltonian if \(X \omega\) is exact.
2. An isotopy \(\phi^t\) of diffeomorphism is called symplectic (respectively, Hamiltonian), if its generating vector fields \(X_t\) are symplectic (respectively, Hamiltonian).

We denote by \(\text{symp}(M, \omega)\) (respectively, \(\text{ham}(M, \omega)\)) the subset of symplectic vector fields (respectively, the subset of Hamiltonian vector fields). Obviously we have the inclusions

\[
\text{ham}(M, \omega) \subset \text{symp}(M, \omega) \subset X(M).
\]

We leave the proof of the following lemma as an exercise.

Lemma 2.3.4 Let \(X, Y\) be two symplectic vector fields. Then the Lie bracket \([X, Y]\) is the Hamiltonian vector field associated with the function

\[
f = -\omega(X, Y).
\]

Proof The proof is left as an exercise. \(\square\)

Proposition 2.3.5

1. Both \(\text{symp}(M, \omega)\) and \(\text{ham}(M, \omega)\) are Lie subalgebras of \(X(M)\).
2. We have

\[
[\text{symp}(M, \omega), \text{symp}(M, \omega)] \subset \text{ham}(M, \omega).
\]

In particular, \([\text{ham}(M, \omega), \text{symp}(M, \omega)] \subset \text{ham}(M, \omega)\), i.e., \(\text{ham}(M, \omega)\) is a normal Lie subalgebra of \(\text{symp}(M, \omega)\).
3. We have the exact sequence of Lie algebras

\[
0 \to \text{ham}(M, \omega) \to \text{symp}(M, \omega) \to H^1(M, \mathbb{R}) \to 0,
\]

where the first map is the inclusion and the second is the map defined by

\[
X \mapsto [X] \omega,
\]

and \(H^1(M, \mathbb{R})\) has the trivial bracket.

Proof Items (1) and (2) are immediate consequences of Lemma 2.3.4. Item (3) is also easy to check from the definitions. We leave the details to the reader. \(\square\)
2.3 The Hamiltonian diffeomorphism group

The group $\text{Symp}(M, \omega)$ is the automorphism group of a geometric structure, the symplectic structure $\omega$, and $\text{symp}(M, \omega)$ is its associated Lie algebra. We will provide $\text{Symp}(M, \omega)$ with the topology induced from the $C^\infty$ topology of $\text{Diff}(M)$, when $M$ is closed.

Proposition 2.3.5 (2) suggests that there should exist a normal subgroup of the Lie group $\text{Symp}(M, \omega)$ whose corresponding Lie algebra is $\text{ham}(M, \omega)$. In fact, there does exist such a subgroup, but its definition is not as natural as that of $\text{Symp}(M, \omega)$, especially when one tries to put a topology on it.

Suppose that $H : [0, 1] \times M \to \mathbb{R}$ is a time-dependent smooth family of functions. We denote by $H_t = H(t, \cdot) : M \to \mathbb{R}$ and by $\phi^t_H$ the flow of the ordinary differential equation

$$\dot{x} = X_{H_t}(x).$$

For simplicity of notation, we will also just write $X_{H_t}(x) = X_H(t, x)$, considering $X_H$ a time-dependent vector field.

**Definition 2.3.6** A diffeomorphism $\phi : M \to M$ is called Hamiltonian if $\phi = \phi^1_H$ with $\phi^0_H = \text{id}$ for a time-dependent Hamiltonian function

$$H : [0, 1] \times M \to \mathbb{R}.$$ 

In this case, we denote $H \mapsto \phi$. We denote by $\text{Ham}(M, \omega)$ the set of Hamiltonian diffeomorphisms.

It is not manifest from the definition that $\text{Ham}(M, \omega)$ forms a group, while it is certainly a subset of $\text{Symp}(M, \omega)$. The proof of this fact involves an interesting algebra of Hamiltonian functions that was introduced by Hofer (H90) and will play an important role in the study of the geometry of Hamiltonian diffeomorphisms and the Floer homology theory.

We start with the following lemma.

**Lemma 2.3.7** Let $h : M \to \mathbb{R}$ be a function and $X_h$ be its associated Hamiltonian vector field. Then, for any symplectic diffeomorphism $\psi : M \to M$, we have

$$\psi^*(X_h) = X_{h \circ \psi}.$$ 

**Proof** We will prove that

$$\psi^*(X_h)|\omega = d(h \circ \psi).$$

(2.3.20)

Since $\psi$ is symplectic and so $\psi^* \omega = \omega$, we have

$$\psi^*(X_h)|\omega = \psi^*(X_h)|\psi^* \omega.$$
On the other hand, the latter is nothing but
\[ \psi^*(X_h) \omega = \psi^*(dh) = d(h \circ \psi). \]
Combining the two, we have proved (2.3.20). \( \square \)

**Proposition 2.3.8** Let \( \phi, \psi \) be two Hamiltonian diffeomorphisms and \( H, K : [0, 1] \times M \to \mathbb{R} \) be their generating Hamiltonians. Then we have the following.

1. \( \overline{H} \mapsto \phi^{-1} \), where \( \overline{H} \) is the Hamiltonian defined by
   \[ \overline{H}(t, x) := -H(t, \phi_H^t(x)). \] (2.3.21)

2. \( H \# K \mapsto \phi \psi \), where \( H \# K : [0, 1] \times M \to \mathbb{R} \) is the Hamiltonian defined by
   \[ H \# K(t, x) := H(t, x) + K(t, (\phi_H^t)^{-1}(x)). \] (2.3.22)

In particular, \( \text{Ham}(M, \omega) \) is a subgroup of \( \text{Symp}(M, \omega) \).

**Proof** Let \( Y_t \) be the vector field generating the isotopy \( (\phi_H^t)^{-1} \), i.e.,
\[ Y_t = \frac{d}{dt}((\phi_H^t)^{-1}) \circ \phi_H^t. \]
We compute
\[
0 = \frac{d}{dt}(id)(x) = \frac{d}{dt} \left( (\phi_H^t)^{-1} \circ \phi_H^t(x) \right) \\
= d(\phi_H^t)^{-1} \left( \frac{d\phi_H^t}{dt}(x) \right) + Y_t \left( (\phi_H^t)^{-1} \circ \phi_H^t(x) \right).
\]
Therefore we have derived
\[
Y_t(x) = -d(\phi_H^t)^{-1} \left( \frac{d\phi_H^t}{dt}(x) \right) = -d(\phi_H^t)^{-1}(X_{\phi_H^t}(x)) \\
= -(\phi_H^t)^*X_{\phi_H^t}(x) = X_{-H \circ \phi_H^t}(x).
\]
From this, we have proved statement (1).

A similar straightforward computation, differentiating the composed isotopy \( t \mapsto \phi_H^t \circ \phi_K^t \), proves statement (2). We leave the details to the reader as an exercise. \( \square \)

**Exercise 2.3.9** Prove that the function (2.3.22) generates the isotopy \( \phi_H^t \circ \phi_K^t \).

Now we will show that \( \text{Ham}(M, \omega) \) is precisely the Lie subgroup of \( \text{Symp}(M, \omega) \) whose associated Lie algebra is given by \( \text{ham}(M, \omega) \). In other words, we need to prove that any one-parameter subgroup \( \{ \phi_s \} \) of \( \text{Ham}(M, \omega) \) is of the form \( \phi_s^h \), which is the flow of the autonomous Hamiltonian vector field associated with a function \( h \in C^\infty(M) \). It turns out that the proof of this fact requires some discussion on the topology of \( \text{Ham}(M, \omega) \) before launching
on its proof, which will involve some non-trivial arguments originally due to Banyaga (Ba78).

### 2.3.2 General topology of \( \text{Ham}(M, \omega) \)

We first give a brief discussion on the \( C^\infty \) topology of \( \text{Symp}(M, \omega) \) induced from that of \( \text{Diff}(M) \). The following local contractibility of the \( C^\infty \) topology is essentially a consequence of Weinstein’s Darboux neighborhood theorem of Lagrangian submanifolds (Wn73), whose proof we give in Section 3.3. We will discuss Lagrangian submanifolds in detail in later chapters.

**Proposition 2.3.10** The group \( \text{Symp}(M, \omega) \) is a closed and locally closed subgroup of \( \text{Diff}(M) \) with respect to the \( C^\infty \) topology. The induced topology of \( \text{Symp}(M, \omega) \) from \( \text{Diff}(M) \) is locally contractible.

Similarly we define the \( C^\infty \) topology of \( \text{Ham}(M, \omega) \) the induced topology from the inclusion \( \text{Ham}(M, \omega) \subset \text{Diff}(M) \). However, unlike in the case of \( \text{Symp}(M, \omega) \), the problem of whether \( \text{Ham}(M, \omega) \) is a closed subgroup of \( \text{Diff}(M) \) or of \( \text{Symp}(M, \omega) \) is highly non-trivial. This question is closely related to the so-called \( C^\infty \) flux conjecture (Ba78) which has been proven by Ono (On06) in full generality. We will discuss the flux homomorphism more in the next section.

We now return to the study of the given one-parameter subgroup \( \phi^s \in \text{Ham}(M, \omega) \).

**Remark 2.3.11** Recall that, for the case of a finite-dimensional Lie group, any one-parameter subgroup automatically defines a smooth isotopy of diffeomorphisms of the Lie group. However, since \( \text{Ham}(M, \omega) \) is an infinite-dimensional group, it is not clear whether a one-parameter subgroup defines a smooth (or even \( C^1 \)) path on \( \text{Ham}(M, \omega) \) in that the curve \( s \mapsto \phi^s(x) \) is a differentiable path on \( M \).

Because of this remark, we will always consider a smooth family \( \{\phi^s\} \subset \text{Ham}(M, \omega) \) in the sense that the family is smooth as a family in \( \text{Diff}(M) \). A natural question to ask then is whether such an isotopy can be obtained by a Hamiltonian flow, i.e., whether there is a time-dependent Hamiltonian \( K : [0, 1] \times M \to \mathbb{R} \) such that

\[
\phi^s = \phi_K^s \circ \phi^0.
\] (2.3.23)

It is useful to give an explicit name to such a family.
**Definition 2.3.12** A (smooth) Hamiltonian path \( \lambda : [0,1] \to \text{Ham}(M,\omega) \) is a map defined by \( \lambda(t)(x) := \Lambda(t,x) \) for a smooth map
\[
\Lambda : [0,1] \times M \to M
\]
such that

1. its derivative \( \dot{\lambda}(t) = \partial \lambda/\partial t \circ (\lambda(t))^{-1} \) is Hamiltonian, i.e., the one-form \( \dot{\lambda}(t) \cdot \omega \) is exact for all \( t \in [0,1] \). We call a function \( H : \mathbb{R} \times M \to \mathbb{R} \) the generating Hamiltonian of \( \lambda \) if it satisfies
\[
\lambda(t) = \phi_H(\lambda(0)) \quad \text{or equivalently} \quad dH_t = \dot{\lambda}(t) \cdot \omega.
\]

2. The diffeomorphism \( \lambda(0) := \Lambda(0,\cdot) : M \to M \) is a Hamiltonian diffeomorphism.

We denote by \( \mathcal{P}^{\text{ham}}(\text{Symp}(M,\omega)) \) the set of Hamiltonian paths \( \lambda : [0,1] \to \text{Symp}(M,\omega) \), and by \( \mathcal{P}^{\text{ham}}(\text{Symp}(M,\omega),\text{id}) \) the set of \( \lambda \) with \( \lambda(0) = \text{id} \). We provide the obvious topology on \( \mathcal{P}^{\text{ham}}(\text{Symp}(M,\omega)) \) and \( \mathcal{P}^{\text{ham}}(\text{Symp}(M,\omega),\text{id}) \) induced by the \( C^\infty \) topology of the corresponding map \( \Lambda \) above.

It is often useful to introduce the inverse of the map \( H \mapsto \phi_H \). We first recall that the constant function generates the constant path and so we need to suitably normalize in order to uniquely define the inverse.

**Definition 2.3.13** Let \((M,\omega)\) be a connected closed symplectic manifold. We say that a function \( h : M \to \mathbb{R} \) is mean-normalized if it satisfies
\[
\int_M h\omega^n = 0. \tag{2.3.24}
\]
We denote by \( C^\infty_m(M) \) the set of mean-normalized smooth functions.

**Remark 2.3.14** We often need to remove the ambiguity of the constant function in relating a Hamiltonian vector field to its associated Hamiltonian function. This is especially the case when one attempts to interpret an invariant constructed out of time-dependent Hamiltonian functions as the one for the corresponding Hamiltonian paths. This normalization will enter into the construction of symplectic invariants through the study of the critical-point theory of the action functional or in the study of Hamiltonian fibrations. We will come back to this normalization problem when we discuss the Floer homology of Hamiltonian periodic orbits and its associated spectral invariants.
The following definition was introduced in \((\text{OhM07})\).

**Definition 2.3.15** Let \(\lambda \in \mathcal{P}^\text{ham}(\text{Symp}(M, \omega), id)\) and \(H\) be the normalized Hamiltonian generating the given Hamiltonian path \(\lambda\), i.e., \(\lambda = \phi_H\). We define two maps

\[
\text{Tan, Dev} : \mathcal{P}^\text{ham}(\text{Symp}(M, \omega), id) \to C^\infty([0, 1] \times M, \mathbb{R})
\]

by the formulae

\[
\text{Tan}(\lambda)(t, x) := H(t, \phi_H^t(x)), \\
\text{Dev}(\lambda)(t, x) := H(t, x)
\]

and call them the *tangent map* and the *developing map*. We call the image of the tangent map the *rolled Hamiltonian* of \(\lambda\) (or \(H\)).

The tangent map corresponds to the map of the tangent vectors of the path. Assigning the usual generating Hamiltonian \(H\) to a Hamiltonian path corresponds to the *developing map* in the Lie group theory: one can ‘develop’ any differentiable path on a Lie group to a path in its Lie algebra using the tangent map and then the right translation.

We will give the proof of the following basic theorem due to Banyaga (\(\text{Ba78}\)) in Section 2.4.

**Theorem 2.3.16** (Banyaga) Any smooth path in \(\text{Diff}(M)\) of Hamiltonian diffeomorphisms is a Hamiltonian path.

Assuming this theorem for the moment, we go back to the study of a smooth one-parameter subgroup \(\phi^s\) in \(\text{Ham}(M, \omega) \subset \text{Diff}(M)\). By this theorem, we know that \(\phi^s\) is a Hamiltonian path, i.e., \(X|\omega\) is exact and so \(X|\omega = dh\) for some smooth function \(h : M \to \mathbb{R}\). We summarize the above discussion as follows.

**Corollary 2.3.17** The Lie algebra of \(\text{Ham}(M, \omega)\) is the set \(\text{ham}(M, \omega)\) consisting of Hamiltonian vector fields.

**Proof** Let \(\phi^s\) be a smooth one-parameter subgroup in \(\text{Ham}(M, \omega) \subset \text{Diff}(M)\). From the group property, the generating vector fields

\[
X_s := \frac{d\phi^s}{ds} \circ (\phi^s)^{-1}
\]
are $s$-independent. We denote the common vector field by $X = X_0$. Now we need to show that $X$ is indeed Hamiltonian, i.e., $X|\omega$ is exact. However, it follows from Theorem 2.3.16 that $\phi^s$ is a Hamiltonian path, i.e., $X|\omega$ is exact and so $X|\omega = dh$ for some smooth function $h : M \to \mathbb{R}$. This finishes the proof. □

Recall the identity $[X_f, X_g] = -X_{\{f, g\}}$ from Lemma 2.3.4 which implies that the assignment

$$f \in C^\infty(M)/\mathbb{R} \mapsto -X_f$$

defines a Lie algebra isomorphism and hence we have the following.

**Proposition 2.3.18** The vector space $C^\infty(M)/\mathbb{R}$ is canonically isomorphic to the Lie algebra $\text{ham}(M, \omega)$ of $\text{Ham}(M, \omega)$ with respect to the Poisson bracket associated with $\omega$.

Now we re-cast the above discussion in the point of view of general Lie group theory. First of all, we have the natural identifications

1. $T_{\text{id}}\text{Ham}(M, \omega) \cong$ the set of Hamiltonian vector fields,
2. the set of Hamiltonian vector fields $\cong$ the set of exact one-forms and
3. $C^\infty_m(M) \cong$ the set of exact one-forms.

Secondly, by the general fact on the Lie group theory, the derivative

$$dL_\psi : T_{\text{id}}\text{Ham}(M, \omega) \to T_\psi \text{Ham}(M, \omega)$$

of the left multiplication (composition)

$$L_\psi : \text{Ham}(M, \omega) \to \text{Ham}(M, \omega); \quad \phi \mapsto \psi \circ \phi \quad (2.3.25)$$

provides an isomorphism between $T_{\text{id}}\text{Ham}(M, \omega)$ and $T_\psi \text{Ham}(M, \omega)$. Note that the conjugation $\phi \mapsto \psi \phi \psi^{-1}$ by a symplectic diffeomorphism $\psi$ on $\text{Ham}(M, \omega)$ is a special case of the adjoint action of the Lie group, $\text{Ham}(M, \omega)$, on its Lie algebra, the set of Hamiltonian vector fields. In terms of the functions in $C^\infty_m(M)$, the induced adjoint action is given by $h \mapsto h \circ \psi^{-1}$.

There is a natural adjoint-invariant norm on $C^\infty(M)/\mathbb{R}$ for any symplectic manifold defined by the assignment

$$\text{osc}(h) := \max_{x \in M} h(x) - \min_{x \in M} h.$$ 

Now the general fact on the Lie group tells us that osc induces a bi-invariant Finsler metric on $T\text{Ham}(M, \omega)$. This is precisely Hofer’s Finsler metric on $\text{Ham}(M, \omega)$. With respect to this metric, we have the identity

$$\int_0^1 ||d\phi'|_{\psi} dt = \int_0^1 ||(d\phi')^{-1} \cdot \phi'|_{\text{id}} dt = \int_0^1 \text{osc}(H_t) dt,$$
where $\phi' = \phi'_H$ and $H : M \times [0, 1] \to \mathbb{R}$ is the normalized Hamiltonian generating the isotopy $\phi'$. It was an amazing insight of Hofer (H90) that the geometry and the dynamics of this invariant metric $\text{Ham}(M, \omega)$ would provide a rich source of the problems of symplectic topology and Hamiltonian dynamics up to the level of $C^0$ topology.

### 2.4 Banyaga’s theorem and the flux homomorphism

The subject presented in this section starts with an attempt to construct an invariant that distinguishes a Hamiltonian diffeomorphism from a general symplectic diffeomorphism lying in $\text{Symp}_0(M, \omega)$, namely the identity component of $\text{Symp}(M, \omega)$. One difficulty of this question lies in the fact that, even when a symplectic diffeomorphism $\psi$ is generated by a non-exact closed one-form, it will not in general imply that $\psi$ is not Hamiltonian.

**Example 2.4.1** Consider the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the symplectic form $\omega = d\theta \wedge d\phi$, where $(\theta, \phi)$ is the obvious coordinate system with $\theta, \phi \in \mathbb{R}/\mathbb{Z}$. Consider the isotopy $\phi^s : T^2 \to T^2$ defined by

$$(\theta, \phi) \mapsto (\theta, \phi + t), \quad 0 \leq t < 1.$$  

This flow is generated by the symplectic vector field associated with the closed one-form $d\theta$, which is not exact: note that $\theta$ is not a real-valued function on $S^1$ but a circle-valued function. However its time one-map $\phi^1 = \text{id}$ is obviously a Hamiltonian diffeomorphism generated by the zero function.

We start with the following fundamental lemma from (Ba78). We provide a somewhat simpler and more direct proof of the lemma than (Ba78).

**Lemma 2.4.2** (Banyaga) Let $\{\phi'_s\}$ be a smooth two-parameter family of diffeomorphisms on $M$. Denote

$$X = X_{s,t} = \frac{\partial \phi'_s}{\partial t} \circ (\phi'_s)^{-1}, \quad Y = Y_{s,t} = \frac{\partial \phi'_s}{\partial s} \circ (\phi'_s)^{-1}.$$  

Then we have

$$\frac{\partial Y}{\partial t} = \frac{\partial X}{\partial s} + [Y, X].$$  

(2.4.26)

**Proof** By the smoothness assumption of the family, by definition, the map

$$(s, t, x) \mapsto \phi'_s(x); \mathbb{R} \times \mathbb{R} \times M \to M.$$
is a smooth map. Fix a point \( x \in M \) and consider the map \( \phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \times M \) defined by
\[
\phi(s, t) = (s, t, \phi'_s(x)).
\]
Obviously the coordinate vector field \( \left\{ \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right\} \) of \( \mathbb{R}^2 \) is \( \phi \)-related to
\[
\left\{ \frac{\partial}{\partial s} \oplus Y_{s,t}, \frac{\partial}{\partial t} \oplus X_{s,t} \right\}.
\]
Therefore we have
\[
\left[ \frac{\partial}{\partial s} \oplus Y_{s,t}, \frac{\partial}{\partial t} \oplus X_{s,t} \right] = 0,
\]
(2.4.27) since \( \left[ \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] = 0 \). By expanding out the bracket, noting that
\[
\left[ \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] = 0 \quad \text{and then varying } x \in M, \text{ we obtain (2.4.26) from (2.4.27).} \quad \square
\]

Let \( \psi \in \text{Symp}_0(M, \omega) \) and \( \psi_t, 0 \leq t \leq 1 \) be a smooth path \( \Psi = \{\psi_t\}_{0 \leq t \leq 1} \) with \( \psi_0 = \text{id}, \psi_1 = \psi \). We consider the generating vector field \( X_t \) and the corresponding one-forms \( \lambda(t) = X_t \| \omega \), and the integral
\[
\Sigma(\psi, \Psi) = \int_0^1 \lambda(t) dt.
\]
Since \( \lambda(t) \) is closed for all \( t \in [0, 1] \), this defines a closed one-form on \( M \).

**Lemma 2.4.3** Let \( \Psi, \Psi' \) be two smooth paths in \( \text{Symp}(M, \omega) \) such that \( \psi_0 = \text{id} = \psi'_0 \) and \( \psi_1 = \psi = \psi'_1 \). If \( \Psi' \) is path-homotopic to \( \Psi \) relative to the ends \( \partial[0, 1] = \{0, 1\} \), then the one-form \( \Sigma(\psi, \Psi) - \Sigma(\psi, \Psi') \) is an exact one-form.

**Proof** Let \( \overline{\Psi} = \{\Psi^s\}_{0 \leq s \leq 1} \) be a homotopy between \( \Psi \) and \( \Psi' \) relative to \( \{0, 1\} \), i.e., satisfying
\[
\Psi^s(0) = \text{id}, \quad \Psi^s(1) = \psi, \quad 0 \leq s \leq 1.
\]
To prove this lemma, it will suffice to prove that the one-form
\[
\frac{d}{ds} \Sigma(\psi, \Psi^s)
\]
is exact. Denoting \( \Psi^s = \{\psi_t^s\}_{0 \leq t \leq 1} \) for \( 0 \leq s \leq 1 \), we define \( X_t^s, Y_t^s \) as in Lemma 2.4.2. Since all \( \psi_t^s \) are symplectic, it follows that \( X_t^s, Y_t^s \) are symplectic vector fields. We now compute
Applying Lemma 2.4.2, we can rewrite this into

$$\frac{d}{ds} \Sigma(\psi, \Psi^s) = \int_0^1 \frac{\partial}{\partial s} \lambda^s(t) dt = \int_0^1 \frac{\partial}{\partial s} (X^s_t) \omega dt = \int_0^1 \frac{\partial X^s_t}{\partial s} \omega dt.$$  

The first term becomes $Y^s_1 \omega - Y^s_0 \omega$, which vanishes for all $s$, since we have $\psi^s_1 \equiv \psi$ and $\psi^s_0 \equiv id$. We also know that the second term is exact because the Lie bracket $[Y^s_t, X^s_t]$ is a Hamiltonian vector field by Proposition 2.3.5 (2). This finishes the proof. □

This lemma enables us to push down the map $\Sigma$ to the universal covering space $\tilde{\text{Symp}}_0(M, \omega)$ and to define a map as follows.

**Definition 2.4.4** We define a map $F_\omega : \tilde{\text{Symp}}_0(M, \omega) \to H^1(M, \mathbb{R})$ by setting

$$F_\omega([\psi, \Psi]) = \left[ \int_0^1 (X_t) \omega dt \right]$$

and call this value the flux of the symplectic path $\Psi$.

We leave the proof of the following lemma as an exercise.

**Exercise 2.4.5** Let $M$ be compact without a boundary. Prove that $F_\omega$ is a surjective group homomorphism with respect to the group structures on $\tilde{\text{Symp}}_0(M, \omega)$ and $H^1(M, \mathbb{R})$.

We recall the short exact sequence

$$0 \to \pi_1(\text{Symp}_0(M, \omega), id) \to \tilde{\text{Symp}}_0(M, \omega) \to \text{Symp}_0(M, \omega) \to 0$$

and that $\pi_1(\text{Symp}_0(M, \omega), id)$ is a countable set.

**Exercise 2.4.6** Prove that $\pi_1(\text{Symp}_0(M, \omega), id)$ is a countable set.

**Definition 2.4.7** (The flux group) The flux group is defined to be the image of $\pi_1(\text{Symp}_0(M, \omega), id)$ in $H^1(M, \mathbb{R})$ under the map $F_\omega$, which we denote by $\text{Flux}(M, \omega)$.

Obviously $\text{Flux}(M, \omega) \subset H^1(M, \mathbb{R})$ is a countable subgroup of $H^1(M, \mathbb{R})$. The importance of this group is illustrated by the following proposition, whose proof is given by Polterovich (Po01).
Proposition 2.4.8  The following statements are equivalent.

(1) Flux($M, \omega$) is a discrete subgroup of $H^1(M, \omega)$.
(2) The subgroup Ham($M, \omega$) $\subset$ Symp$_0(M, \omega)$ is $C^\infty$-closed.
(3) Ham($M, \omega$) is locally contractible.

The following conjecture was first raised by Banyaga (Ba78) as a question and proved by Ono (On06). It has commonly been called the flux conjecture.

Theorem 2.4.9 (Ono (On06))  Let ($M, \omega$) be a closed symplectic manifold. Then the group Flux($M, \omega$) is discrete.

The proof of this theorem is highly non-trivial. It involves a sophisticated version of the Floer homology theory of symplectic fixed points. We refer the interested reader to the original article (On06) for this proof. Various special cases had been proven earlier. (See (Ba78) and (LMP98) for example.)

There is an elegant geometric description of Flux($M, \omega$), which is useful for the study of the flux group. Let $\gamma : S^1 \to$ Symp$_0(M, \omega)$ be a loop. We would like to describe $F_\omega(\gamma) \in H^1(M, \mathbb{R})$ by its values under the paring

$$H^1(M, \mathbb{R}) \times H_1(M, \mathbb{Z}) \to \mathbb{R}.$$  

Let $a \in H_1(M, \mathbb{Z})$ and $\alpha$ be any one-cycle satisfying $[\alpha] = a$. When we are given the loop $\gamma = \{\psi_t\}_{0 \leq t \leq 1}$, we define a two-cycle denoted by $\gamma \cdot \alpha$ as follows. Write $\alpha = \sum_j k_j c_j$, where $c_j : [0, 1] \to M$ are one-chains. Then we define a two-chain $\gamma \cdot \alpha$ by the formula

$$\gamma \cdot \alpha = \sum_j k_j C_j,$$

where $C_j : S^1 \times [0, 1] \to M$ is defined by $C_j(t, s) = \psi_t(c_j(s))$. It is easy to check that $\gamma \cdot \alpha$ defines a two-cycle if $\alpha$ is a one-cycle, and, if $\alpha$ and $\alpha'$ are homologous to each other, then so are $\gamma \cdot \alpha$ and $\gamma \cdot \alpha'$. We denote the common homology class of the two cycles by $\gamma \cdot a$. We then have the following proposition, whose proof is a straightforward calculation.

Proposition 2.4.10  Let $a \in H_1(M, \mathbb{Z})$ and $\gamma \cdot a \in H_2(M, \mathbb{Z})$ as above. Then we have

$$\langle F_\omega(\gamma), a \rangle = \langle \omega, \gamma \cdot a \rangle.$$  

(2.4.28)

Proof  We leave the proof to readers as an exercise.  \( \square \)
By definition, the map $F_\omega$ induces a canonical homomorphism

$$f_\omega : \text{Symp}_0(M, \omega) \to \frac{H^1(M, \mathbb{R})}{\text{Flux}(M, \omega)}.$$ 

Now we have the following commutative diagram of exact sequences:

$$
\begin{array}{ccccccccc}
0 & \to & \pi_1(\ker F_\omega) & \to & \pi_1(\text{Symp}_0(M, \omega), id) & \to & \text{Flux}(M, \omega) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \ker F_\omega & \to & \text{Symp}_0(M, \omega) & \xrightarrow{F_\omega} & H^1(M, \mathbb{R}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \ker f_\omega & \to & \text{Symp}_0(M, \omega) & \xrightarrow{f_\omega} & \frac{H^1(M, \mathbb{R})}{\text{Flux}(M, \omega)} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & 
\end{array}
$$

The following is the fundamental theorem on the structure of $\text{Ham}(M, \omega)$ proven by Banyaga (Ba78).

**Theorem 2.4.11** (Banyaga)

1. $\ker f_\omega = \text{Ham}(M, \omega)$.
2. $\text{Ham}(M, \omega)$ is a simple group and
   $$[\text{Symp}_0(M, \omega), \text{Symp}_0(M, \omega)] = \text{Ham}(M, \omega).$$

Since the proof of (2) is highly non-trivial and does not seem to allow much simplification from (Ba78), we refer the reader to (Ba78) for the complete proof. Here we will focus on (1) only and give its complete proof.

**Proof of (1).** The inclusion $\text{Ham}(M, \omega) \subset \ker f_\omega$ immediately follows from the definition: indeed we have $\text{Ham}(M, \omega) \subset \ker F_\omega \subset \ker f_\omega$. For the opposite inclusion, let $\psi \in \ker f_\omega \subset \text{Symp}_0(M, \omega)$. By definition, we can pick an isotopy $\Psi = \{\psi_t\}_{0 \leq t \leq 1} \subset \text{Symp}_0(M, \omega)$ so that the closed one-form $\text{Flux}_\omega(\{\psi_t\}_{0 \leq t \leq 1})$ has its cohomology class in $\text{Flux}(M, \omega) \subset H^1(M, \mathbb{R})$. In other words, there exists a loop $h : S^1 \to \text{Symp}_0(M, \omega)$ based at the identity such that

$$\text{Flux}_\omega(\{|\Psi|\}) = \text{Flux}_\omega(h),$$
or, equivalently,

$$\text{Flux}_\omega(\Psi \circ h^{-1}) = 0. \quad (2.4.29)$$

Note that we still have $$\Psi \circ h^{-1}(0) = id, \, \Psi \circ h^{-1}(1) = \psi.$$ Now the following lemma proves that $$\psi \in \text{Ham}(M, \omega),$$ which will prove that $$\ker f_\omega \subset \text{Ham}(M, \omega)$$ and hence follows the proof of (1).

**Lemma 2.4.12** (Proposition II.3.3 of (Ba78))  
Suppose that a smooth path $$\Psi' : [0, 1] \rightarrow \text{Symp}_0(M, \omega)$$ with $$\Psi'(0) = id$$ has zero flux, i.e., $$\text{Flux}_\omega(\Psi') = 0.$$ Then $$\Psi'(1) = \psi$$ is contained in $$\text{Ham}(M, \omega).$$

**Proof**  We will deform the path $$\Psi'$$ to a path $$\Psi^1$$ without changing the end points so that $$\Psi^1$$ becomes a Hamiltonian path, i.e., so that the form $$X^1_t \omega$$ is exact for all $$0 \leq t \leq 1,$$ where the vector field $$X^1_t$$ is given by

$$X^1_t := \frac{\partial \psi^1_t}{\partial t} \circ (\psi^1_t)^{-1}.$$ 

First note that, by reparameterizing the path $$\Psi'$$ without changing the flux, we may assume that $$\frac{\partial \psi'_t}{\partial t} = 0$$ near $$t = 0, 1.$$ We denote $$X'_t = \frac{\partial \psi'_t}{\partial t} \circ (\psi'_t)^{-1}$$ and the corresponding one-form by $$\alpha'_t = X'_t \omega.$$ By virtue of this choice, we know that $$\alpha'_t = 0$$ for $$t$$ near $$0, 1.$$ These forms are all closed, and we have

$$\int_0^1 \alpha'_t \, dt = 0 \quad (2.4.30)$$

by the hypothesis of zero flux. We consider one-forms $$\{\alpha^s_t\}_{0 \leq s \leq 1}$$ defined by

$$\alpha^s_t = (1 - s)\alpha'_{(1-s)t} - \int_0^{1-s} \alpha'_u \, du,$$

for each $$s \in [0, 1].$$ Obviously all these forms are closed. It follows from (2.4.30) and the boundary flatness of $$\alpha'_t$$ that we have

$$\alpha^0_t = \alpha'_t, \quad \alpha^1_t \equiv 0. \quad (2.4.31)$$

Furthermore, we also derive

$$\int_0^1 \alpha^s_t \, dt = 0, \quad (2.4.32)$$

for all $$s \in [0, 1].$$ Now we denote by $$X^s_t$$ the symplectic vector fields associated with $$\alpha^s_t$$ and by $$\Psi^s = \{\psi^s_t\}_{0 \leq s \leq 1}$$ the symplectic isotopy generated by $$X^s = \{X^s_t\}_t.$$ It follows from (2.4.32) that $$F_\omega(\Psi^s) = 0$$ for all $$s \in [0, 1].$$ In particular, $$f_\omega(\psi^s_t) = 0$$ for all $$s \in [0, 1]$$ by the definition of $$f_\omega : \text{Symp}_0(M, \omega) \rightarrow$$
2.4 Banyaga’s theorem and the flux homomorphism

\( H^1(M, \mathbb{R})/\text{Flux}(M, \omega) \). Now consider the isotopy \( \Phi = \{ \phi_t \}_{0 \leq t \leq 1} \) with \( \phi_t := \psi^{1-t}(1) \), \( 0 \leq t \leq 1 \). Note that the second equation of (2.4.31) implies that

\[ \psi^1_t = \text{id} \]

and, in particular, \( \phi_0 = \psi_1^{} = \text{id} \). From the first equation of (2.4.31), we have \( \psi^0_t = \psi \) and hence \( \phi_1 = \psi^0_1 = \psi_1' \). Therefore \( \Phi \) is a symplectic path from the identity to \( \psi_1' \) satisfying

\[ f_\omega(\phi_t) \equiv 0 \]

for all \( 0 \leq t \leq 1 \). Furthermore, it is easy to see that the path \( \Phi \) is homotopic to \( \Psi' \), on noting that

\[ \psi^s_0 = \psi_1^{} = \text{id}, \quad s, t \in [0, 1]. \]

Now the proof will be finished by the following general lemma. \( \square \)

**Lemma 2.4.13** Let \( \Phi = \{ \phi_t \} \) be a symplectic path with \( \phi_0 = \text{id} \) such that

\[ f_\omega(\phi_t) = 0, \quad t \in [0, 1]. \]

Then \( \Phi \) is a Hamiltonian path.

**Proof** We choose a two-parameter family \( \{ \phi^{s}_t \}_{0 \leq s, t \leq 1} \) such that

\[ \phi^0_t = \text{id}, \quad \phi^1_t = \phi_t. \]

For example, the family \( \phi^{s}_t := \phi_{st} \) will serve the purpose. As before, we denote

\[ X^s_t = \frac{\partial \phi^s_t}{\partial t}, \quad Y^s_t = \frac{\partial \phi^s_t}{\partial s}. \]

For each \( t \in [0, 1] \), using Lemma 2.4.2, we compute

\[ X_t] \omega = X^1_t] \omega = \int_0^1 \frac{\partial X^s_t}{\partial s} \, ds = \int_0^1 \left( \frac{\partial Y^s_t}{\partial t} + [X^s_t, Y^s_t] \right) \omega \, ds. \tag{2.4.33} \]

Here the second term is exact since \([\text{symp}, \text{symp}] \subset \text{ham}\). For the first term, we have

\[ \int_0^1 \frac{\partial Y^s_t}{\partial t} \omega \, ds = \frac{\partial}{\partial t} \int_0^1 (Y^s_t] \omega) \, ds. \]

On the other hand, for each fixed \( t \), \( Y^s_t \) is the vector field generating the path \( s \mapsto \phi^s_t, s \in [0, 1] \). Hence

\[ \int_0^1 Y^s_t] \omega \, ds = F_\omega(\phi^1_t, \{ \phi^s_t \}_{0 \leq s \leq 1}) = F_\omega(\phi_t, \{ \phi^s_t \}_{0 \leq s \leq 1}). \]
By hypothesis, \( \phi_t \in \ker f_{\omega} \), i.e.,
\[
F_{\omega}(\phi_t, \{\phi^s_t\}_{0 \leq s \leq 1}) \in \text{Flux}(M, \omega).
\]
Obviously the assignment \( t \mapsto F_{\omega}(\phi_t, \{\phi^s_t\}_{0 \leq s \leq 1}) ; [0, 1] \to \text{Flux}(M, \omega) \subset H^1(M, \mathbb{R}) \) is continuous. On the other hand, the subset \( \text{Flux}(M, \omega) \subset H^1(M, \mathbb{R}) \) is a countable subset and so the assignment must be constant. Hence we conclude that
\[
\frac{\partial}{\partial t} \left[ \int_0^1 (Y^s_t) \omega ds \right] \equiv 0,
\]
i.e., the one-forms \( (\partial/\partial t) \int_0^1 (Y^s_t) \omega ds \) are exact for all \( t \in [0, 1] \). This proves that the first term of (2.4.33) is also exact.

On substituting the two terms into (2.4.33), we conclude that \( X_t \omega \) are exact for all \( t \in [0, 1] \), which finishes the proof of the lemma. \( \square \)

This also finishes the proof of (1) of Theorem 2.4.11. \( \square \)

### 2.5 Calabi homomorphisms on open manifolds

So far we have considered \( \text{Ham}(M, \omega) \) on closed symplectic manifolds \((M, \omega)\). In this section, we summarize the structure of \( \text{Ham}(M, \omega) \) on open symplectic manifolds, i.e., \((M, \omega)\) which is either non-compact or compact with a nonempty boundary \( \partial M \).

We denote the set of compactly supported (in \( \text{Int } M \)) symplectic diffeomorphisms by \( \text{Symp}_c(M, \omega) \subset \text{Diff}_c(M, \omega) \). Next we define the support of a Hamiltonian function \( H = H(t, x) \) by
\[
\text{supp}(H) = \bigcup_{t \in [0, 1]} \text{supp}(H_t).
\]

**Definition 2.5.1** We say that a smooth path \( \lambda : [0, 1] \to \text{Symp}_c(M, \omega) \) is a compactly supported Hamiltonian path if \( \lambda = \phi_H \) for a Hamiltonian function \( H : [0, 1] \times M \to \mathbb{R} \) such that \( \text{supp } H \) is compact and \( \text{supp } H \subset \text{Int}(M) \). We define
\[
\mathcal{P}^{\text{ham}}(\text{Symp}_c(M, \omega), \text{id})
\]
to be the set of such \( \lambda \) with \( \lambda(0) = \text{id} \). A compactly supported symplectic diffeomorphism \( \phi \) is a **compactly supported Hamiltonian diffeomorphism** if \( \phi = \text{ev}_1(\lambda) \) for some \( \lambda \in \mathcal{P}^{\text{ham}}(\text{Symp}_c(M, \omega), \text{id}) \). We denote the set of compactly supported Hamiltonian diffeomorphisms by
Recall that the symplectic form $\omega$ induces the Liouville measure on $M$ by integrating the volume form

$$\Omega = \frac{1}{n!} \omega^n.$$ 

We consider the integral

$$\text{Cal}(H) = \int_0^1 \int_M H_t \Omega \wedge dt$$

for a compactly supported Hamiltonian $H$. Denote $H := \text{Dev}(\lambda)$.

**Lemma 2.5.2** Suppose that $\lambda$ and $\lambda'$ are two Hamiltonian paths with $\lambda(0) = id = \lambda'(0)$ and $\lambda(1) = \lambda'(1)$, and $\lambda \sim \lambda'$. Suppose that $\text{Dev}(\lambda) = H$ and $\text{Dev}(\lambda') = H'$. Then we have $\text{Cal}(H) = \text{Cal}(H')$.

**Proof** Let $\lambda^s$ with $s \in [0, 1]$ be a homotopy relative to the end points with $\lambda^0 = \lambda$ and $\lambda^1 = \lambda'$. Consider the one-parameter family of Hamiltonians $H^s = \text{Dev}(\lambda^s)$ and let $\{H^s_t\}$ be the associated two-parameter family of compactly supported smooth functions on $M$. By the reparameterization of $s$ by $\rho : [0, 1] \to [0, 1]$ with $\rho(s) \equiv 0$ near $s = 0$ and $\rho(s) \equiv 1$ near $s = 1$ which flattens the end of $H^s$, we may always assume that $H^s$ is end-flat. We also denote by $F : [0, 1]^2 \times M \to \mathbb{R}$ the Hamiltonian generating the $s$-Hamiltonian isotopy $(s, t) \mapsto \lambda^s(t)$. From the end-flatness of $H^s$, it follows that $F(s, t, x) \equiv 0$ near $s = 0, 1$.

Using the support condition $\text{supp} H \subset \text{Int} M$, we derive from (2.4.26)

$$\frac{\partial H}{\partial s} = \frac{\partial F}{\partial t} - [H, F].$$

(See Section 18.3 for a detailed discussion of this derivation.) Now we compute

$$\text{Cal}(H') - \text{Cal}(H) = \int_0^1 \int_M H'_t d\mu dt - \int_0^1 \int_M H_t d\mu dt$$

$$= \int_0^1 \int_0^1 \int_M \frac{\partial H^s}{\partial s} d\mu dt ds$$

$$= \int_0^1 \int_0^1 \int_M \left( \frac{\partial F}{\partial t} - [H, F] \right) d\mu dt ds.$$
For the first term we obtain
\[
\int_0^1 \int_0^1 \int_M \frac{\partial F}{\partial t} \, d\mu \, dt \, ds = \int_0^1 \int_M \int_0^1 \frac{\partial F}{\partial t} \, dt \, d\mu \, ds
= \int_0^1 \int_M (F(s, 1, x) - F(s, 0, x)) \, d\mu \, ds = 0.
\]

On the other hand, we recall the formula
\[
\{H, F\} \, d\mu = \mathcal{L}_{X_F}(H) \omega^n \frac{n!}{n!} = \mathcal{L}_{X_F}(H) \omega^n \frac{n!}{n!} = \mathcal{L}_{X_F}(H) \omega^n \frac{n!}{n!} = \frac{d(X_F) H \omega^n}{n!},
\]
where the last identity follows from \(d(H \omega^n) = 0\). Hence we obtain
\[
\int_M \{H, F\} \, d\mu = \int_M \frac{d(X_F) H \omega^n}{n!} = \int_{\partial M} \frac{(X_F) H \omega^n}{n!} = 0,
\]
since \(F^s\) is supported in \(\text{Int} \, M\). This finishes the proof of \(\text{Cal}(H') = \text{Cal}(H)\). \(\square\)

An immediate corollary of this lemma is the following construction of a non-trivial homomorphism on \(\tilde{\text{Ham}}_c(M, \omega)\).

**Proposition 2.5.3** The map \(\text{Cal}^\text{path} : \mathcal{P}\text{ham}(\text{Symp}_0(M, \omega), id) \to \mathbb{R}\) defined by
\[
\text{Cal}^\text{path}(\lambda) = \text{Cal}(\text{Dev}(\lambda))
\]
defines a non-trivial homomorphism, which depends on the path homotopy class of \(\lambda\) relative to the end points and hence induces a homomorphism on \(\tilde{\text{Ham}}_c(M, \omega)\). We call \(\text{Cal}^\text{path}\) the Calabi homomorphism of the path.

**Proof** For the homomorphism property, we compute
\[
\text{Cal}^\text{path}(\lambda \cdot \lambda') = \text{Cal}(\text{Dev}(\lambda \cdot \lambda'))
= \int_0^1 \int_M (H_t + H'_t \circ (\phi_H^t)^{-1}) \Omega \, dt
= \int_0^1 \int_M (H_t + H'_t) \Omega \, dt = \text{Cal}^\text{path}(\lambda) + \text{Cal}^\text{path}(\lambda').
\]
Here the second identity follows from the composition formula
\[
(H \# H')_t = H_t + H'_t \circ (\phi_H^t)^{-1}
\]
and the third from \((\phi_H^t)^* \Omega = \Omega\).

Next we recall that the universal covering space \(\tilde{\text{Ham}}_c(M, \omega)\) is realized by the path homotopy classes \([\lambda]\) relative to end points.

Then the statement of the proposition is just a translation of the above lemma. \(\square\)
Corollary 2.5.4  Denote by \( \widehat{\text{Cal}} : \widehat{\text{Ham}}_c(M, \omega) \to \mathbb{R} \) the descendent of Cal\(^{\text{path}}\). Then \( \ker \widehat{\text{Cal}} \) is a proper normal subgroup of \( \widehat{\text{Ham}}_c(M, \omega) \).

In fact Banyaga (Ba78) proved that \( \ker \widehat{\text{Cal}} \) is a simple group when \( M \) is connected.

In general the homomorphism \( \widehat{\text{Cal}} \) cannot be pushed down to a homomorphism on \( \text{Ham}_c(M, \omega) \) itself, unless \( \text{Ham}_c(M, \omega) \) is simply connected. The last condition holds, for example, when \( M = D^{2n} \).

2.5.1 On exact symplectic manifolds

There is one special case for which \( \text{Cal} \) can be pushed down to \( \text{Ham}_c(M, \omega) \): this the case when \( (M, \omega) \) is exact.

Suppose that \( \omega \) is exact and \( \omega = d\alpha \). We note that \( \beta := \phi^* \alpha - \alpha \) is a closed one-form with \( \beta \equiv 0 \) near \( \partial M \). Let \( \phi \in \text{Ham}_c(M, \omega) \), and let \( \lambda \) be a Hamiltonian path given by \( \lambda(t) = \phi^t_H \) with \( \lambda(0) = id, \lambda(1) = \phi^1_H = \phi \). Consider the family

\[
(\phi^t)^* \alpha - \alpha, \quad \phi^t = \phi^t_H.
\]

This is obviously compactly supported as \( \phi^t = id \) near \( \partial M \). We compute

\[
\frac{d}{dt}((\phi^t)^* \alpha - \alpha) = (\phi^t)^*(\mathcal{L}_{\phi_t}(\alpha))
\]

\[
= d((\phi^t)^*(X_H|\alpha) + (\phi^t)*X_H|d\alpha)
\]

\[
= d\left((\phi^t)^*(X_H|\alpha) + (\phi^t)^*(H_t)\right).
\]

On the other hand, we have a useful formula that can be proved by a straightforward calculation.

Lemma 2.5.5

\[
(\phi^t)^*(X_H|\alpha) \land \omega^n = (\phi^t)^*(nH_t \omega^n - d(nH_t \alpha \land \omega^{n-1})). \tag{2.5.35}
\]

Making these substitutions, we derive

\[
\int_M (\phi^* \alpha - \alpha) \land \alpha \land \omega^{n-1} = \int_M \left(\int_0^1 \frac{d}{dt}((\phi^t)^* \alpha - \alpha) dt\right) \land \alpha \land \omega^{n-1}
\]

\[
= \int_0^1 \left(\int_M d\left((\phi^t)^*(X_H|\alpha) + (\phi^t)^*(H_t)\right) \land \alpha \land \omega^{n-1}\right) dt
\]

\[
= \int_0^1 \left(\int_M \left((\phi^t)^*(X_H|\alpha) + (\phi^t)^*(H_t)\right) d\alpha \land \omega^{n-1}\right) dt.
\]
= \int_0^1 \int_M ((\phi^t)^*(X_H) \alpha) + (\phi^t)^* (H_t) \wedge \omega^n \wedge dt

= (n + 1) \int_0^1 \int_M (\phi^t)^* H_t \omega^n \wedge dt = (n + 1) \text{Cal}(\lambda),

where we substitute (2.5.35) into the fifth equality.

We summarize the above discussion in the following theorem.

**Theorem 2.5.6** (Banyaga (Ba78)) Suppose that $\omega = d\alpha$. Let $\phi \in \text{Ham}_c(M, \omega)$, and let $\lambda$ be any Hamiltonian path such that $\lambda(1) = \phi$. Then we have

$$\text{Cal}^\text{path}(\lambda) = \frac{1}{n + 1} \int_M (\phi^* \alpha - \alpha) \wedge \alpha \wedge \omega^{n-1}. \tag{2.5.36}$$

In particular, $\text{Cal}^\text{path}(\lambda)$ depends only on the final point $\lambda(1) = \phi$ and hence defines a non-trivial homomorphism

$$\text{Cal} : \text{Ham}_c(M, d\alpha) \to \mathbb{R}.$$ 

In particular, $\text{Ham}_c(M, d\alpha)$ is not a simple group.

**Remark 2.5.7** On general $(M, \omega)$, we have only the homomorphism of the type

$$\text{Cal} : \text{Ham}_c(M, \omega) \to \mathbb{R}/\Lambda,$$

which is still surjective where $\Lambda \subset \mathbb{R}$ is the set

$$\Lambda = \{\tilde{\text{Cal}}(g) \mid g \in \pi_1(\text{Ham}_c(M, \omega))\}.$$ 

We have just shown this above for when $\omega$ is exact, $\Lambda = \{0\}$. In any case, we conclude that $\text{Ham}_c(M, \omega)$ is not simple in general.

### 2.5.2 An example: the case of the two-disc

Consider the case of the two-disc $D^2$ with the standard symplectic form $\omega = dx \wedge dy = rdr \wedge d\theta$ in polar coordinates $(r, \theta)$. Since $\text{Ham}(D^2, \partial D^2) = \text{Symp}(D^2, \partial D^2) = \text{Diff}^\omega(D^2, \partial D^2)$, which is contractible, we have a non-trivial homomorphism $\text{Cal} : \text{Ham}(D^2, \partial D^2) \to \mathbb{R}$. Here the group $G(D^2, \partial D^2)$ denotes the set of relevant diffeomorphisms supported in the interior of the disc $D^2$ for each of the three cases of $G$.

We now give interesting examples of the sequences of Hamiltonian diffeomorphisms (or equivalently area-preserving diffeomorphisms in this case).

We have the following important formula for the behavior of Calabi invariants on $D^2$ under the Alexander isotopy.
Corollary 2.5.8 Let \( \lambda \) be a given Hamiltonian path on \( D^2 \) and let \( \lambda_a \) be the map defined by
\[
\lambda_a(t, x) = \begin{cases} 
  a\lambda(t, x/a), & \text{for } |x| \leq a(1 - \eta), \\
  x, & \text{otherwise}, 
\end{cases}
\]
for \( 0 < a \leq 1 \). Then \( \lambda_a \) satisfies
\[ \text{Cal}^\text{path}(\lambda_a) = a^4 \text{Cal}^\text{path}(\lambda). \tag{2.5.37} \]

Proof A straightforward calculation proves that \( \lambda_a \) is generated by the (unique) Hamiltonian defined by
\[
\text{Dev}(\lambda_a)(t, x) = \begin{cases} 
  a^2 H(t, x/a), & \text{for } |x| \leq a(1 - \eta), \\
  0, & \text{otherwise}, 
\end{cases}
\]
where \( H = \text{Dev}(\lambda) \). From this, we derive the formula
\[
\text{Cal}^\text{path}(\lambda_a) = \int_0^1 \int_{D^2(a(1-\eta))} a^2 H(t, \frac{x}{a}) \Omega \wedge dt
\]
\[
= a^4 \int_0^1 \int_{D^2} H(t, y) \Omega \wedge dt = a^4 \text{Cal}^\text{path}(\lambda).
\]
This proves (2.5.37). \( \square \)

With these preparations, we consider the set of dyadic numbers \( 1/2^k \) for \( k = 0, \ldots \). Let \( (r, \theta) \) be polar coordinates on \( D^2 \). Then the standard area form is given by
\[ \omega = r \, dr \wedge d\theta. \]
Consider maps \( \phi_k : D^2 \rightarrow D^2 \) of the form given by
\[ \phi_k = \phi_{\rho_k} : (r, \theta) \rightarrow (r, \theta + \rho_k(r)), \]
where \( \rho_k : (0, 1) \rightarrow [0, \infty) \) is a smooth function supported in \( (0, 1) \). It follows that \( \phi_{\rho_k} \) is an area-preserving map generated by an autonomous Hamiltonian given by
\[ F_{\phi_k}(r, \theta) = -\int_1^r s\rho_k(s) \, ds. \]
Therefore its Calabi invariant becomes
\[ \text{Cal}(\phi_k) = -\int_{D^2} \left( \int_1^r s\rho_k(s) \, ds \right) r \, dr \, d\theta = \pi \int_0^1 r^3 \rho_k(r) \, dt. \tag{2.5.38} \]
We now choose $\rho_k$ in the following way.

(1) $\rho_k$ has support in $1/2^k < r < 1/2^{k-1}$.

(2) For each $k = 1, \ldots$, we have

$$\rho_k(r) = 2^4 \rho_{k-1}(2r) \quad (2.5.39)$$

for $r \in (1/2^k, 1/2^{k-1})$.

(3) $\text{Cal}(\phi_1) = 1$.

Since the $\phi_k$ have disjoint supports by construction, we can freely compose them without concerning ourselves about the order of composition. It follows that the infinite product

$$\Pi_{k=0}^\infty \phi_k$$

is well defined and defines a continuous map that is smooth except at the origin, at which $\phi_\rho$ is continuous but not differentiable. This infinite product can also be written as the homeomorphism having its values given by $\phi_\rho(0) = 0$ and

$$\phi_\rho(r, \theta) = (r, \theta + \rho(r)),$$

where the smooth function $\rho : (0, 1] \to \mathbb{R}$ is defined by

$$\rho(r) = \rho_k(r) \quad \text{for} \quad \left[ \frac{1}{2^k}, \frac{1}{2^{k-1}} \right], \quad k = 1, 2, \ldots.$$

It is easy to check that $\phi_\rho$ is smooth $D^2 \setminus \{0\}$ and is a continuous map, even at 0, which coincides with the above infinite product. Obviously the map $\phi_{-\rho}$ is the inverse of $\phi_\rho$, which shows that it is a homeomorphism. Furthermore, we have

$$\phi_\rho^\star(r \, dr \wedge d\theta) = r \, dr \wedge d\theta \quad \text{on} \quad D^2 \setminus \{0\},$$

which implies that $\phi_\rho$ is indeed area-preserving. We now define another sequence of area-preserving diffeomorphisms $\psi_k$ by

$$\psi_k = \prod_{i=1}^k \phi_k.$$

We summarize the above discussion of the example in the following proposition. The existence of such a sequence has been known since the work by Gambaudo and Ghys in (GG04).

**Proposition 2.5.9** Consider $\phi_k$ and $\psi_k$ defined above.

(1) $\phi_k \to \text{id}$ uniformly but $\text{Cal}(\phi_k) = 1$ for all $k$.

(2) $\psi_k$ uniformly converges to an area-preserving homeomorphism that is not differentiable at the origin and $\text{Cal}(\psi_k) \not\to \infty$. 
An immediate corollary of this is the following classical fact

**Corollary 2.5.10** (Compare this with (GG04))  
Consider the homomorphism $\text{Cal} : \text{Diff}^\Omega(D^2, \partial D^2) \to \mathbb{R}$.

1. The map $\text{Cal}$ is not continuous in $C^0$ topology of $\text{Diff}^\Omega(D^2, \partial D^2)$.
2. The homomorphism $\text{Cal}$ is not bounded in the $C^0$ topology, i.e., there exists a sequence $\psi_k \in \text{Diff}^\Omega(D^2, \partial D^2)$ with $d_{C^0}(\psi_k, \text{id}) \leq C$ but $\text{Cal}(\psi_k) \searrow \infty$.

This phenomenon turns out to be closely tied to the symplectic nature of the two-dimensional area-preserving dynamics. We will return to this important aspect of the two-dimensional dynamics in Chapter 6 and Section 22.6.
In this chapter, we introduce the notion of Lagrangian submanifolds as the natural boundary condition of the least action principle. Following the physicists’ terminology, we will call the resulting equation arising from the least action principle the equation of motion of the given functional in general.

To motivate our discussion, we start with the least action principle on the cotangent bundle $M = T^*N$ with its canonical symplectic form.

### 3.1 The conormal bundles

On the cotangent bundle $T^*N$, the Hamiltonian action functional on $T^*N$ is defined as the integral

$$\mathcal{A}_H(\gamma) = \int_{\gamma} \theta - \int_0^1 H(t, \gamma(t)) dt.$$  

It is customary to write $\theta = p \, dq$, which we may also adopt from time to time. We will derive the first variation formula of $\mathcal{A}_H$. By considering a variation $\xi$ along the given path $\gamma$ that is supported in a small open subset of $[0, 1]$ whose image is contained in $T^*U$, where $U \subset N$ is a coordinate neighborhood, we may use calculations in canonical coordinates. Then we have $\theta = \sum p_j \, dq^j$ and so $\mathcal{A}_H$ has precisely the same form as (1.3.26) for the Hamiltonian functional on the classical phase space. In the more formal expression, (1.3.26) is written as

$$d\mathcal{A}_H(\gamma)(\xi) = \langle \theta(\gamma(1)), \xi(1) \rangle - \langle \theta(\gamma(0)), \xi(0) \rangle + \int_0^1 \omega_0(\dot{\gamma} - X_H(\gamma), \xi) dt.$$  

(3.1.1)

**Exercise 3.1.1**  Prove this formula.
Here we recall that in our convention the canonical symplectic form $\omega_0$ on the contangent bundle is given by $\omega_0 = -d\theta$.

In the classical phase space, besides the periodic boundary condition, we looked at the two-point boundary condition as the natural boundary condition, i.e., as a boundary condition that kills the boundary terms in (3.1.1). This amounts to restricting $A_H$ to the space of paths

$$P(T^*N; T^*_xN, T^*_yN) = \{ \gamma : [0, 1] \to T^*N \mid \gamma(0) \in T^*_xN, \gamma(1) \in T^*_yN \}$$

for the given two configuration points $x, y \in N$. It turns out that this natural boundary condition allows a more general class than the two-point boundary condition, which we call the conormal boundary condition.

**Definition 3.1.2 (Conormal bundles)** Let $S \subset N$ be a smooth submanifold. The conormal bundle of $S$ is defined by

$$\nu^*S = \{ \alpha \in T^*_qN \mid q \in S, \alpha|_{T_qS} = 0 \}.$$

The following lemma is the basic property of conormal bundles, which demonstrates that they are the natural boundary condition for the action functional $A_H$ in that this condition kills the boundary contribution from (3.1.1).

**Lemma 3.1.3** Let $S \subset N$ be a smooth submanifold and $i_S : \nu^*S \subset T^*N$ be its conormal bundle. Then we have $i_S^*\theta = 0$. In particular, it satisfies $i_S^*\omega_0 = 0$. Furthermore, $\dim \nu^*S = n (= \dim N)$ for any $S \subset N$.

**Proof** Let $\xi \in T_{\alpha}(\nu^*S)$. Then $d\pi(\xi) \in T_{\pi(\alpha)}S$. Therefore we have

$$\theta_{\alpha}(\xi) = \alpha(d\pi(\xi)) = 0$$

since $\alpha \in \nu^*_{\pi(\alpha)}S$. \qed

The conormal boundary condition includes the following two particular cases as the special cases.

**Example 3.1.4**

1. Let $S = \{q\}$ be a point in $N$. Then $\nu^*\{q\} = T^*_qN$ is a fiber of $T^*N$ at $q$.
2. When $S = N$, $\nu^*S = o_N$, the zero section.

Now let $S_0, S_1 \subset N$ be two submanifolds and $\nu^*S_i, i = 0, 1$ be their conormal bundles. Consider the subset

$$L(\nu^*S_0, \nu^*S_1) = \{ \gamma : [0, 1] \to T^*N \mid \gamma(0) \in \nu^*S_0, \gamma(1) \in \nu^*S_1 \}$$
of paths on $T^*N$. Denote by $d\mathcal{A}_{(H; S_0, S_1)}$ the restriction of $d\mathcal{A}_H$ to $L(v^*S_0, v^*S_1)$. Then we have

$$d\mathcal{A}_{(H; S_0, S_1)}(\gamma)(\xi) = \int_0^1 \omega(\dot{\gamma} - X_H(t, \gamma(t)), \xi(t)) dt$$

and hence the critical point set $\text{Crit}(\mathcal{A}_{(H; S_0, S_1)})$ has a one-to-one correspondence to the set of Hamiltonian trajectories from $v^*S_0$ to $v^*S_1$, i.e., those paths $z$ satisfying

$$\dot{z} = X_H(t, z), \ z(0) \in v^*S_0, \ z(1) \in v^*S_1.$$

There is no direct analog to the action functional $\mathcal{A}_H$ on general symplectic manifolds $(M, \omega)$ since its definition uses the one-form $\theta$ in an essential way. To regard the Hamilton equation above as an extremal path for some functional on the path space of $M$, we first need to develop a fair amount of background geometry of a general symplectic manifold. We start with some basic symplectic linear algebra.

### 3.2 Symplectic linear algebra

In this section, we closely follow the exposition given by Weinstein (Wn87).

Let $(S, \Omega)$ be a finite-dimensional symplectic vector space. The linear map $\widetilde{\Omega} : S \to S^*$ defined by

$$\widetilde{\Omega}(v)(w) := \Omega(v, w)$$

is a skew-symmetric isomorphism. Let $C \subset S$ be a subspace. We define the $\Omega$-orthogonal complement, denoted by $C^\Omega$,

$$C^\Omega := \{ v \in S \mid \Omega(v, w) = 0 \ \forall \ w \in C \}.$$

**Definition 3.2.1** Let $C \subset S$ be a subspace.

1. $C$ is called **isotropic** if $C \subset C^\Omega$.
2. $C$ is called **coisotropic** if $C \supset C^\Omega$.
3. $C$ is called **Lagrangian** if $C = C^\Omega$.

We now show that when $C$ is coisotropic the quotient $C/C^\Omega$ carries a symplectic inner product canonically induced from $\Omega$. The following is a fundamental lemma that is the linear version of the so-called *symplectic reduction*.

**Lemma 3.2.2** Let $(V, \Omega)$ be a symplectic vector space and $C \subset V$ be a coisotropic subspace. Let $C^\Omega \subset C$ be its null subspace. Denote by $\iota_C : C \subset V$
and \( \pi_C : C \rightarrow C/C^\Omega \) the natural inclusion and the projection. Then \( \Omega \) induces the canonical symplectic bilinear form \( \omega_C \) on the quotient \( C/C^\Omega \), which is characterized by the relation

\[
\iota_C^* \Omega = \pi^* \omega_C. \tag{3.2.2}
\]

**Proof** By the definition of \( C^\Omega \), we have

\[
\iota_C^* \Omega|_{C^\Omega} \equiv 0,
\]

and so \( \iota_C^* \Omega \) descends to a skew-symmetric two-form on the quotient \( C/C^\Omega \).

It remains to show its nondegeneracy. Let \( v \in C \) with \( \pi_C(v) \neq 0 \), i.e., \( v \notin C^\Omega \). We should find a vector \( w \in C \) such that \( \Omega(v, w) \neq 0 \). But this is obvious from the definition of \( C^\Omega \) since \( v \notin C^\Omega \). This finishes the proof. \( \Box \)

### 3.2.1 The Lagrangian Grassmannian

**Definition 3.2.3** Denote by \( \mathcal{L}(S) \) the set of Lagrangian subspaces of \((S, \Omega)\), and \( Sp(S, \Omega) \subset GL(S) \) the subgroup of automorphisms of \((S, \Omega)\).

**Proposition 3.2.4**

1. Every isotropic subspace is contained in a Lagrangian subspace.
2. \( Sp(S, \Omega) \) acts transitively on \( \mathcal{L}(S) \).
3. Every Lagrangian subspace has a Lagrangian complement.

**Proof** We will leave the statements (1) and (2) as exercises (or see \((Ar67)\) for their proof). We will give only the proof of (3). Choose any two complementary subspaces \( U_1, U_2 \) of \( L \), not necessarily Lagrangian, i.e., \( U_1, U_2 \) satisfy

\[
S = U_1 \oplus L = U_2 \oplus L.
\]

Then noting that the set of complements, denoted by \( \text{Comp}_L(S) \), to \( L \) forms an affine space modeled by \( \text{Hom}(S/L, L) \), we can take the ‘difference’ of \( U_1 \) and \( U_2 \), which we denote by

\[
\delta_L(U_1, U_2) : S/L \to L.
\]

More precisely, if we write \( q \in S/L \) as \( q = L + v \) for some \( v \in S \), then there exists a unique intersection \( p_j, j = 1, 2 \), of \( q \cap U_j \). We define

\[
\delta_L(U_1, U_2)(q) = p_2 - p_1. \tag{3.2.3}
\]

We leave the following properties of \( \delta_L \) as exercises.
Exercise 3.2.5  Prove the following.

(1) \( \delta_L(U, U) = 0. \)
(2) \( \delta_L \) satisfies a ‘cocycle condition’
\[
\delta_L(U_1, U_2) + \delta_L(U_2, U_3) = \delta_L(U_1, U_3),
\]
and in particular \( \delta_L(U_1, U_2) = -\delta_L(U_2, U_1). \)
(3) Given a complement \( U_1 \) and a map \( \delta \in \text{Hom}(S/L, L) \), there exists a unique complement \( U_\delta \) such that \( \delta_L(U_1, U_\delta) = \delta \). In that case, we denote
\[
U_\delta = U_1 + \delta.
\]
(4) \( U \) is Lagrangian if and only if \( \delta_L(U, U^\Omega) = 0. \)

Now the proof of statement (3) of Proposition 3.2.4 will be finished by the following general lemma.

Lemma 3.2.6  Let \( L \subset S \) be a Lagrangian subspace of \((S, \Omega)\) and \( C \) be any complement, not necessarily Lagrangian. Then the following statements hold.

(1) The \( \Omega \)-orthogonal complement, \( C^\Omega \), is also a complement, and \( C \) is Lagrangian if and only if \( C = C^\Omega \).
(2) Denote by \( \iota_L : \text{Comp}_L(S) \to \text{Comp}_L(S) \) the involution defined by \( \iota_L(C) = C^\Omega \). Then we have
\[
\text{Fix } \iota_L = \mathcal{L}_L(S),
\]
where \( \mathcal{L}_L(S) \) is the subset of \( \mathcal{L}(S) \) consisting of Lagrangian complements of \( L \).
(3) Let \( U_1, U_2 \in \text{Comp}_L(S) \), and let \( \delta_L(U_1, U_2) \in \text{Hom}(S/L, L) \) be their difference. Then we have
\[
\delta_L(U_1^\Omega, U_2^\Omega) = \delta_L(U_1, U_2).
\]
(4) The ‘average’ \((C + C^\Omega)/2\) is Lagrangian. Moreover, the map
\[
C \mapsto \frac{C + C^\Omega}{2}; \quad \text{Comp}_L(S) \to \mathcal{L}_L(S)
\]
defines a canonical projection to the Lagrangian complement, and the projection as a map from \( \text{Comp}_L(S) \to \text{Comp}_L(S) \) is smooth.

Proof  Since \( C \) is a complement to \( L \), we have \( C \oplus L = S \). Therefore we have \( 0 = (C \oplus L)^\Omega = C^\Omega \cap L^\Omega \). But since \( L \) is Lagrangian, we have \( L = L^\Omega \) and hence we have proved \( C^\Omega \cap L = 0 \). Since \( \dim L = \dim C^\Omega = n \), this implies \( C^\Omega \oplus L = S \), which finishes the proof of the first statement. The proof of statement (2) is immediately evident from this.
For the proof of statement (3), we note that for any Lagrangian subspace $L \subset S$, for each given $\ell \in L$, the interior product $\ell \lrcorner \Omega$ defines a linear functional on $S$ that is restricted to zero on $L$. Therefore it defines a linear map $I_L : L \rightarrow (S/L)^*$, which is an isomorphism. This isomorphism induces a natural identification
$$\delta \mapsto I_L \circ \delta; \text{Hom}(S/L, L) \cong \text{Hom}(S/L, (S/L)^*)$$.

The latter space is precisely the space of quadratic forms on $S/L$.

With this having been said, by the nondegeneracy of $\Omega$, statement (3) is equivalent to the equality
$$\Omega(\delta_L(U_1, U_2)v, v) = -\Omega(\delta_L(U_1^\Omega, U_2^\Omega)v, v) \quad (3.2.7)$$
for all $v \in S$. Let $v$ be any element of $S$ and denote by $p_i \in U_i \cap (L + v)$ and $p_i^\Omega \in U_i^\Omega \cap (L + v)$ the unique intersections for $i = 1, 2$. By definition, we have
$$\delta_L(U_1, U_2) = p_2 - p_1, \quad \delta_L(U_1^\Omega, U_2^\Omega) = p_2^\Omega - p_1^\Omega.$$ 

Therefore we need to prove
$$\Omega(p_2 - p_1, v) \Omega(p_2^\Omega - p_1^\Omega, v) = 0. \quad (3.2.8)$$

By the choice of $p_i \in U_i$ and $p_i^\Omega \in U_i^\Omega$, we have $\Omega(p_i^\Omega, p_i) = 0$. Since $p_i, p_j^\Omega \in L + v$ for all $i, j = 1, 2$, and $L$ is Lagrangian,
$$\Omega(p_i^\Omega - v, p_j - v) = 0 \quad (3.2.9)$$
for all $i, j = 1, 2$. This equation gives rise to
$$\Omega(p_i^\Omega - p_j, v) = \Omega(p_i^\Omega, p_j) \quad (3.2.10)$$
for all $i, j = 1, 2$. Now, using (3.2.10), we compute
$$\Omega(p_2^\Omega - p_1^\Omega, v) - \Omega(p_2 - p_1, v) = \Omega(p_2^\Omega - p_1^\Omega - p_2 + p_1, v)$$
$$= \Omega(p_2^\Omega - p_2, v) - \Omega(p_1^\Omega - p_1, v)$$
$$= \Omega(p_2^\Omega, p_2) - \Omega(p_1^\Omega, p_1)$$

Here we used (3.2.10) for the third equality. The last two terms vanish since $p_i \in U_i$ and $p_i^\Omega \in U_i^\Omega$ for $i = 1, 2$. This proves (3.2.8) and hence Statement (3).

Finally we turn to the last statement. By the first step, we know that both $C$ and $C^\Omega$ lie in the affine space $\text{Comp}_L(S)$ and so the average $M := (C + C^\Omega)/2$
makes sense as an element of $\text{Comp}_L(S)$. More specifically, the average $M$ of $C$ and $C^\Omega$ in $\text{Comp}_L(S)$ is uniquely determined by the equation

$$\delta_L(M, C) + \delta_L(M, C^\Omega) = 0.$$ 

Now it suffices to prove that $M$ is a fixed point of the involution $\iota_L: \text{Comp}_L(S) \rightarrow \text{Comp}_L(S)$. On the other hand, statement (3) gives rise to

$$\delta_L(M^\Omega, C) + \delta_L(M^\Omega, C^\Omega) = \delta_L(M, C^\Omega) + \delta_L(M, C) = 0.$$ 

Therefore $M^\Omega$ is also the average of $C$ and $C^\Omega$. By the uniqueness of the average, we conclude that $M = M^\Omega$ and so $M$ must be Lagrangian by statement (4) of Exercise 3.2.5. This finishes the proof. □

Statement (4) of this lemma then proves statement (3) of Proposition 3.2.4. □

The proofs of the above immediately give rise to the following corollaries.

**Corollary 3.2.7** Let $L \subset S$ be a Lagrangian subspace, with $\delta$ and $I_L$ as above. Let $C \subset S$ be a Lagrangian complement to $L$ and let $U \in \text{Comp}_L(S)$ be a complement. Denote $U = C + \delta$, where $\delta \in \text{Hom}(S/L, L) \cong \text{Hom}(S/L, (S/L)^*)$. Then $U$ is Lagrangian if and only if $\delta$ is symmetric, i.e., $\delta = \delta^\ast$.

**Proof** The quadratic form associated with $U$ is nothing but

$$\Omega(\delta_L(C, U)v, v), \ [v] \in S/L$$

in the above proof. Equation (3.2.7) and $C$ being Lagrangian implies

$$\Omega(\delta_L(C, U)v, v) + \Omega(\delta_L(C, U^\Omega)v, v) = 0.$$ 

By the skew-symmetry of $\Omega$, we can rewrite this into

$$\Omega(\delta_L(C, U)v, v) = \Omega(v, \delta_L(C, U^\Omega)v).$$

In other words, we have

$$\delta_L(C, U)^\ast = \delta_L(C, U^\Omega)$$

as a quadratic form in $\text{Hom}(S/L, (S/L)^*)$. Therefore, if $U$ is Lagrangian, i.e., $U = U^\Omega$, then $\delta_L(C, U)^\ast = \delta_L(C, U)$, i.e., $\delta_L(C, U)^\ast$ is symmetric. Conversely, suppose that $\delta_L(C, U)^\ast$ is symmetric. Then the last equation implies

$$0 = -\delta_L(C, U) + \delta_L(C, U^\Omega) = \delta_L(U, C) + \delta_L(C, U^\Omega) = \delta_L(U, U^\Omega).$$
Here we used the cocycle condition in Exercise 3.2.5 for the last two identities. It follows from Exercise 3.2.5 (3) that this equation $\delta_L(U, U^\Omega) = 0$ implies that $U$ is Lagrangian. This finishes the proof.

This provides the following fundamental theorem on the structure of the set of Lagrangian subspaces of a symplectic vector space. We denote by $\mathcal{L}(S) = \mathcal{L}(S, \Omega)$ the set of Lagrangian subspaces. It is often called the Lagrangian Grassmannian.

**Theorem 3.2.8** Let $\dim S = 2n$. $\mathcal{L}(S; \Omega)$ is a nonempty manifold of dimension $n(n + 1)/2$ modeled by the space of symmetric matrices of rank $n$. In fact, the set of Lagrangian complements to any given Lagrangian subspace $L \subset S$ is canonically an affine space modeled by the set $S^2(S/L)$ of symmetric quadratic forms on $S/L$, which in particular is contractible.

### 3.2.2 Arnol’d stratification

For this purpose, we need some preparation on the structure of the Lagrangian Grassmannian $\mathcal{L}(S, \Omega)$. Let $L \subset S$ be a given Lagrangian subspace and consider the stratification of $\mathcal{L}(S, \Omega)$,

$$\mathcal{L}(S, \Omega) \supset \mathcal{L}_1(S, \Omega; L) \supset \cdots \supset \mathcal{L}_n(S, \Omega; L) = \{L\},$$

where $\mathcal{L}_k(S, \Omega; L)$ is the subset of $\mathcal{L}(S, \Omega)$ defined by

$$\mathcal{L}_k(S, \Omega; L) = \{V \in \mathcal{L}(S, \Omega) \mid \dim(V \cap L) \geq k\}$$

for $k = 1, \ldots, n$. This stratification, which was introduced by Arnol’d (Ar67), forms the basis of the analysis of the Lagrangian Grassmannian $\mathcal{L}(S, \Omega)$. We call this stratification Arnol’d’s stratification with respect to $L$.

It follows from Theorem 3.2.8 that each $\mathcal{L}_k(S, \Omega; L)$ is a closed subset of $\mathcal{L}(S, \Omega)$ and its open stratum has codimension $k(k + 1)/2$ in $\mathcal{L}(S, \Omega)$. In particular, for $k = 1$, Arnol’d (Ar67) proved the following basic fact on $\mathcal{L}_1(S, \Omega; L)$, which we call the Maslov cycle associated with $L \subset S$.

**Proposition 3.2.9** Let $L \in \mathcal{L}(S, \Omega)$. Then $\mathcal{L}_1(S, \Omega; L)$ satisfies the following properties.

1. It is of codimension 1 in $\mathcal{L}(S, \Omega)$.
2. It is co-oriented in that there exists a vector field $\xi$ along $\mathcal{L}_1(S, \Omega; L)$ such that $\xi$ is transverse to $\mathcal{L}_1(S, \Omega; L)$.
3. The subset $\mathcal{L}_1(S, \Omega; L) \setminus \mathcal{L}_2(S, \Omega; L)$ is a smooth manifold and $\mathcal{L}_2(S, \Omega; L)$ has codimension 2 in $\mathcal{L}_1(S, \Omega; L)$. 


Proof Properties (1) and (3) follow immediately from the dimension formula
\[ \text{codim } L_k(S, \Omega; L) = \frac{k(k+1)}{2}. \]

For the proof of (2), we choose a Darboux basis
\[ \{e_1, \ldots, e_n, f_1, \ldots, f_n\} \]
such that \( \text{span}\{f_1, \ldots, f_n\} = L \) and identify \( L \subset (S, \Omega) \) with \( \sqrt{-1} \mathbb{R}^n \subset \mathbb{C}^n \) and \( \text{span}\{e_1, \ldots, e_n\} \) with \( \mathbb{R}^n \) by the symplectic linear transformation
\[ \Phi : (S, \Omega) \to \mathbb{C}^n; \quad e_j \mapsto \frac{\partial}{\partial x_j}, \quad f_k \mapsto \frac{\partial}{\partial y_k}, \]
where \( z_j = x_j + \sqrt{-1}y_j \) is the standard complex coordinates of \( \mathbb{C}^n \).

For each Lagrangian subspace \( \lambda \subset \mathbb{C}^n \), consider the smooth Lagrangian path defined \( \theta \mapsto e^{i\theta} \cdot \lambda \). This induces a vector field \( \xi \) on \( \Lambda(n) := L(\mathbb{R}^{2n}, \omega_0) \) by the formula
\[ \xi(\lambda) = \frac{d}{d\theta}\bigg|_{\theta=0} e^{i\theta} \cdot \lambda. \]

We restrict \( \xi \) to the Maslov cycle \( \Lambda_1(n) := L_1(\mathbb{R}^{2n}, \omega_0; i\mathbb{R}^n) \) associated with \( i\mathbb{R}^n \), i.e.,
\[ \Lambda_1(n) := \{ \lambda \in \Lambda(n) \mid \text{dim } \lambda \cap i\mathbb{R}^n \geq 1 \}. \]

Let \( \lambda = A \cdot \mathbb{R}^n \in \Lambda_1(n) \setminus \Lambda_2(n) \) for a unitary matrix \( A \). Then \( A \cdot \mathbb{R}^n \cap i\mathbb{R}^n \) has dimension 1. Let \( 0 \neq v \in \mathbb{R}^n \) satisfy \( Av \in i\mathbb{R}^n \). With the decomposition \( A = A_{re} + iA_{im} \), the relation \( Av \in i\mathbb{R}^n \) is equivalent to
\[ A_{re} v = 0, \quad (3.2.11) \]
or equivalently \( v \in \ker A_{re} \). Now we consider \( e^{i\theta} \cdot \lambda = e^{i\theta} A \cdot \mathbb{R}^n \).

Lemma 3.2.10 We have
\[ \ker(e^{i\theta} A)_{re} = \{0\} \]
for \( \theta \) with \( 0 < |\theta| \leq \epsilon \) for any sufficiently small \( \epsilon > 0 \).

Proof Because the nullity of a matrix is upper semi-continuous or never drops for a continuous family of matrices, it suffices to prove that for any non-zero vector satisfying (3.2.11), \( v \notin \ker(e^{i\theta} A)_{re} \) for any \( \theta \) with \( 0 < |\theta| \) sufficiently small. But we note that
\[ (e^{i\theta} A)_{re} = \cos \theta A_{re} - \sin \theta A_{im}, \quad (e^{i\theta} A)_{im} = \sin \theta A_{re} + \cos \theta A_{im}. \]
For any \( v \) satisfying (3.2.11), we obtain
\[ (e^{i\theta} A)_{re} v = \cos \theta A_{re} v - \sin \theta A_{im} v = -\sin \theta A_{im} v \neq 0 \]
for any \( \theta \) with \( 0 < |\theta| < \pi \). This finishes the proof. \[ \square \]
This lemma implies that $e^{i\theta} \cdot \lambda = (e^{i\theta}A) \cdot \mathbb{R}^n$ lies in $\Lambda^0(n) := \Lambda(n) \setminus \Lambda_1(n)$ for all such $\theta \neq 0$.

In particular, we can express the Lagrangian subspace $\lambda(\theta) := e^{i\theta} \cdot \lambda = U(\theta) \cdot \mathbb{R}^n$ with $U(\theta) = e^{i\theta}A$ in the form of

$$\lambda(\theta) = \mathbb{R}^n + iS(\theta) \cdot \mathbb{R}^n = (I + iS(\theta)) \cdot \mathbb{R}^n$$

for some symmetric $n \times n$ matrix $S(\theta)$. Furthermore, Corollary 3.2.7 implies that $\lambda'(0)$ is transversal to $\Lambda_1(n)$ if and only if $S'(\theta)$ is invertible.

Now we compute $S'(\theta)$. Since $(U(\theta))_{re} = (e^{i\theta}A)_{re}$ is invertible and $U(\theta)_{im}$ does not have eigenvalue $-1$ for $\theta$ with $0 < |\theta| < \pi$, we can express $S(\theta)$ as

$$S = U_{re}^{-1}U_{im} = -i(U + \overline{U})^{-1}(U - \overline{U}) = -i(I + U^{-1}\overline{U})^{-1}(I - U^{-1}\overline{U})$$

$$= -i(I + \overline{U}U)^{-1}(I - \overline{U}U).$$

Set $D(\theta) = \overline{U}U$ and then we have

$$S = -i(I + D)^{-1}(I - D).$$

Note that $D$ is the unitary symmetric matrix which depends only on the Lagrangian subspace $\lambda(\theta)$. As long as $\theta \neq 0$, this expression is well defined and smooth in $\theta$. Since $D(\theta) = e^{-2i\theta}A A^\dagger$, it follows that $D'(\theta) = -2iD(\theta)$. Using this, we compute

$$S'(\theta) = -4D(\theta)^2(I + D(\theta))^{-1} = -(I + S(\theta)^2).$$

This is a negative definite matrix whose eigenvalues are less than or equal to $-1$. This finishes the proof of statement (2) of Proposition 3.2.9.

In particular, the Maslov cycle $L_1(S, \Omega; L)$ defines a canonical cohomology class $\mu = \mu(S, \Omega; L)$ in $H^1(\mathcal{L}(S, \Omega), \mathbb{Z})$ that is independent of the choice of reference Lagrangian $L$.

**Exercise 3.2.11** Prove that the cohomology class $\mu(S, \Omega; L)$ does not depend on the choice of $L$.

The above proof also shows that the tangent space $T_L\mathcal{L}(S, \Omega)$ is canonically isomorphic to the set $S^2(L)$ of symmetric quadratic forms on $S/L$.

In fact, we have the following lemma.

**Lemma 3.2.12** The tangent cone $T_{L_1(S, \Omega)} \subset T_L\mathcal{L}(S, \Omega)$ decomposes the tangent space $T_L\mathcal{L}(S, \Omega) \setminus \{0\}$ into $n + 1$ disjoint open cones.

**Proof** It follows from Corollary 3.2.7 that the tangent space $T_L\mathcal{L}(S, \Omega)$ is canonically isomorphic to the set $S^2(L)$ of symmetric quadratic forms and
$T_L\mathcal{L}(S, \Omega) \setminus T_L\mathcal{L}_1(S, \Omega)$ corresponds to nondegenerate quadratic forms. Furthermore, two nondegenerate quadratic forms with fixed signature can be connected by a continuous path in the corresponding cone. This finishes the proof. □

Under the isomorphism $\xi \in T_L\mathcal{L}(S, \Omega) \mapsto Q_\xi \in S^2(L)$, we have the decomposition

$$T_L\mathcal{L}(S, \Omega) \setminus T_L\mathcal{L}_1(S, \Omega) = \bigcup_{k=0}^{n} C_k(L),$$

where $C_k(L)$ is the open cone given by

$$C_k(L) = \{ \xi \in T_L\mathcal{L}(S, \Omega) \mid Q_\xi \text{ has index } k \}.$$

An immediate corollary of the lemma is the following.

**Corollary 3.2.13** There exists a neighborhood $U$ of $L \in \mathcal{L}(S, \Omega)$ such that the set $U \setminus \mathcal{L}_1(S, \Omega; L)$ has exactly $n + 1$ connected components, each of which contains $L$ in its closure.

We denote by $U_k(L) = U \cap C_k(L)$ the component associated with $C_k(L)$ given above.

Using this corollary, we prove the following proposition.

**Proposition 3.2.14** Let $(S, \Omega)$ be a symplectic vector space and $L, V_1 \subset (S, \Omega)$ be two Lagrangian subspaces of $S$ with $L \cap V_1 = \{0\}$. Consider smooth paths $\alpha : [0, 1] \to \mathcal{L}(S, \Omega)$ satisfying

1. $\alpha(0) = L, \alpha(1) = V_1$
2. $\alpha(t) \in \mathcal{L}_0(S, \Omega; L) := \mathcal{L}(S, \Omega) \setminus \mathcal{L}_1(S, \Omega; L)$ for all $0 < t \leq 1$
3. $\alpha'(0) \in C_0(L) \subset T_L\mathcal{L}(S, \Omega)$.

Then any two such paths $\alpha_1, \alpha_2$ are homotopic to each other relative to the end.

**Proof** Let $\alpha_1, \alpha_2 : [0, 1] \to \mathcal{L}(S, \Omega)$ be two such paths. We consider the concatenated loop $\alpha := \alpha_1 * \alpha_2^{-1}$ defined by

$$\alpha_1 * \alpha_2^{-1} = \begin{cases} \alpha_1(2t), & t \in [0, \frac{1}{2}], \\ \alpha_2(2(1-t)), & t \in [\frac{1}{2}, 1]. \end{cases}$$
By the conditions (1) and (2), this loop does not intersect the Maslov cycle $\mathcal{L}_1(S, \Omega; L)$ away from $t = 0 \equiv 1$. On the other hand, Condition (3) implies that there exists some $\epsilon_0 > 0$ such that $\alpha([-\epsilon_0, 0) \cup (0, \epsilon_0])$ is contained in the conical neighborhood $U_n(L)$. Therefore we can push $\alpha$ away from $L$ into $U_n(L) \subset \mathcal{L}(S, \Omega)$ on $[-\epsilon, \epsilon]$ without altering $\alpha$ on $S^1 \setminus [-2\epsilon, 2\epsilon]$. Now the deformed loop does not intersect the Maslov cycle $\mathcal{L}_1(S, \Omega; L)$ at all, and hence it can be contracted to the constant path $V_1$ relative to $V_1$. This proves that $\alpha_0$ is homotopic to $\alpha_1$ relative to the end points, and hence the proof has been completed. □

We refer the reader to (Ar67), (GS77) or (RS93) for further elegant discussion on the topology of $\mathcal{L}(S, \Omega)$ and the description of the Maslov index on $\mathcal{L}(S, \Omega)$. We will get to the discussion on the Maslov index on the Lagrangian Grassmannian in the later chapters when we talk about pseudoholomorphic discs and the Floer theory of Lagrangian intersections. In the meantime, we return to the proof of the Darboux–Weinstein theorem.

### 3.3 The Darboux–Weinstein theorem

Consider the Lagrangian submanifolds $L \subset (M, \omega)$ on general symplectic manifolds.

**Definition 3.3.1** Let $(M, \omega)$ be a symplectic manifold of dimension $2n$, and let $L$ be a manifold of dimension $n = \frac{1}{2} \dim M$. An immersion (respectively embedding) $i : L \to M$ is called Lagrangian immersion (respectively Lagrangian embedding) if $i^*\omega = 0$. When $i$ is an embedding, we call its image an (embedded) Lagrangian submanifold.

The first general theorem on Lagrangian submanifolds is the following neighborhood theorem of Weinstein (Wn73).

**Theorem 3.3.2 (The Darboux–Weinstein theorem)** Let $j : L \to (M, \omega)$ be a compact Lagrangian immersion and let $Z : L \to T^*L$ be the zero section. Then there exists an immersion $E : V \to M$

from a neighborhood of $Z$ in $T^*L$ such that

1. $E^*\omega = \omega_0$
2. $E \circ Z = j$.

If $j$ is an embedding in addition, then $E$ can be made an embedding.
We will give the proof of this theorem by reducing it to Weinstein’s deformation lemma, Lemma 2.2.22. This involves the study of basic symplectic linear algebra developed in Section 3.2. Here we closely follow the exposition given by Weinstein (Wn87).

### 3.3.1 Proof of the Darboux–Weinstein theorem

We first give an outline of the proof.

1. **Step 1.** Use the canonical symplectic algebra developed in the previous subsection and solve the problem of finding the diffeomorphism in the linearized level. More precisely, we will find a bundle map $A : T^*L \to TM$ over the inclusion $j : L \to M$ so that the bundle map
   \[
   dj \oplus A : TL \oplus T^*L \to j^*TM
   \]
   becomes a symplectic bundle map over $L$, i.e., so that it becomes symplectic over the fibers. This symplectic map then can be composed with the canonical isomorphism
   \[
   T(T^*L)|_{o_L} \cong TL \oplus T^*L
   \]
   so that it defines a symplectic bundle map
   \[
   dj \oplus A : T(T^*L)|_{o_L} \to j^*(TM)
   \]
   over the identity on $L$ when $o_L$ is identified with $L$.

2. **Step 2.** We choose a map $E_0 : V_0 \to M$, where $V_0$ is a neighborhood of $o_L \subset T^*L$, such that
   \[
   dE_0|_{o_L} = dj \oplus A.
   \]
   By choosing $V_0$ sufficiently small, $E_0$ becomes immersed.

3. **Step 3.** We compare the canonical symplectic form $\omega_0$ on $T^*L$ with $E_0^*\omega$ on $V_0 \subset T^*L$. By the choice of $E_0$,
   \[
   \omega_0|_{o_L} = E_0^*\omega|_{o_L}.
   \]
   Now we apply the deformation lemma, Lemma 2.2.22 to find a diffeomorphism $\psi : V \to V_0$ so that $\psi^*(E_0^*\omega) = \omega_0$ on $V$ and $\psi|_{o_L} = id_{o_L}$. Then the immersion $E = E_0 \circ \psi$ will do our purpose.

   It remains to carry out the proof of the first step. Note that both $TL \oplus T^*L$ and $j^*TM$ are symplectic vector bundles over $L$. We have the following lemma.
Lemma 3.3.3 Let \((S, \Omega)\) be a symplectic vector space. Let \(S = L \oplus L'\), where \(L, L'\) are Lagrangians. Then the map \(B : L' \to L^*\) by \(B(v) = \widetilde{\Omega}(v)|_L\) is an isomorphism and the direct sum map \(1 \oplus B : S = L \oplus L' \to L \oplus L'\) is a symplectic isomorphism.

Proof The proof is left as an exercise. \(\square\)

Since \(j : L \to M\) is a Lagrangian immersion, \(TL \subset j^*TM\) is a Lagrangian subbundle over \(L\). We denote by \(\Omega\) the fiberwise symplectic form on the bundle \(j^*TM\). We choose any complementary smooth subbundle \(U \subset j^*TM\), i.e., \(U\) satisfying \(TL \oplus U = j^*TM\). It follows from Lemma 3.2.6 that \(U^\Omega\) is another complementary subbundle that is smooth. Then we take the average \(C = (U + U^\Omega)/2\) which defines a Lagrangian subbundle. Here the smoothness of \(U^\Omega\) or \(C\) follows from the functorial construction of the average carried out in Section 2.3.8. Thus we have the Lagrangian splitting \(j^*TM = TL \oplus C\).

Applying Lemma 3.3.3 to \(C\) and \(T^*L\), we produce a bundle isomorphism \(B : C \to T^*L\). Now we set \(A = B^{-1}\) and then \(1 \oplus A : TL \oplus T^*L \to TL \oplus C = j^*TM\) is a symplectic bundle isomorphism, which finishes the proof of Step 1, and hence the proof of the Darboux–Weinstein theorem.

We now state the following parameterized version of the Darboux–Weinstein theorem, whose proof we leave as an exercise.

Exercise 3.3.4 (The parameterized Darboux–Weinstein theorem) Let \(\psi : [0, 1] \times L \to M\) be a Lagrangian isotopy, i.e., a smooth map such that each \(\psi_t : L \to M\) is a Lagrangian embedding. Then there exists a smooth map

\[ \Phi : \cup_{t \in [0, 1]} V_t \subset [0, 1] \times T^*L \to M \]

such that \(\Phi_t : V_t \subset T^*L \to M\) is a symplectic embedding satisfying

1. \(\Phi_t^*\omega = \omega_0\)
2. \(\Phi_t \circ Z = \psi_t\) on \(L\) for all \(t \in [0, 1]\).

In particular, if \(L\) is compact, we can choose a neighborhood

\[ \mathcal{V} \subset \overline{\mathcal{V}} \subset \cap_{t \in [0, 1]} V_t \]

and \(\Phi_t^*\omega = \omega_0\) on \(\mathcal{V}\) for all \(t \in [0, 1]\).
3.4 Exact Lagrangian submanifolds

The conormal bundles on the cotangent bundle are very special cases of the Lagrangian submanifolds which can be defined on general symplectic manifolds.

Before launching into the general study of Lagrangian submanifolds, let us look at some examples.

Example 3.4.1 Consider the cotangent bundle $M = T^*N$ with its canonical symplectic form.

(1) The conormal bundles $\nu^*S$: for these, we have

$$i^*\theta \equiv 0.$$  \hspace{1cm} (3.4.13)

(2) The graphs of closed one-forms: let $\beta$ be a closed one-form on $N$ and consider the $\beta$ as the associated section map $\tilde{\beta} : N \rightarrow T^*N$ and its image denoted by $\text{Graph} \beta$. In this case, it follows from (2.1.2) that

$$d(\tilde{\beta}^*\theta) = d\beta = 0.$$  \hspace{1cm} (3.4.14)

In particular, the graph of the differential $df$ of a smooth function $f : N \rightarrow \mathbb{R}$ is Lagrangian.

The graphs of the exact one-forms $df$ are particular cases of the so-called exact Lagrangian submanifolds.

Definition 3.4.2 A symplectic manifold $(M, \omega)$ is called exact if $\omega = d\alpha$ for a one-form $\alpha$ on $M$.

On exact symplectic manifolds $\omega = d\alpha$, it follows that $i : L \rightarrow M$ is Lagrangian if and only if $i^*\alpha$ is closed.

Definition 3.4.3 A Lagrangian submanifold $i : L \rightarrow M$ in an exact symplectic manifold $(M, d\alpha)$ is called exact if $i^*\alpha$ is exact.

One immediate consequence of the definition of exactness is that there exists a function $f : L \rightarrow \mathbb{R}$ such that $i^*\alpha = df$. The choice of $f$ is unique up to addition of constants. We will see later that it is important to regard the pair $(L, f)$ with $df = i^*\alpha$ as an exact Lagrangian submanifold in the variational study of the action functional.
Another basic property of exact Lagrangian submanifolds is the following vanishing result of the *periods*, which is an immediate consequence of Stokes’ theorem.

**Proposition 3.4.4** Let \( L \subset (M, \omega) \) be an exact Lagrangian submanifold with \( \omega = d\alpha \). Suppose that \( w : (\Sigma, \partial\Sigma) \to (M, L) \) is any smooth map from a compact two-dimensional surface \( \Sigma \) with boundary \( \partial\Sigma \). Then we have

\[
\int_{\Sigma} w^* \omega = 0.
\]

**Proof** Using \( \omega = d\alpha \) and applying Stokes’ formula, we derive

\[
\int_{\Sigma} w^* \omega = \int_{\Sigma} w^* d\alpha = \int_{\partial\Sigma} w^* \alpha.
\]

Then exactness of \( L \) implies \( i_L^* \alpha = df \) for some function \( f \) on \( L \). Therefore we obtain \( w^* \alpha = d(w^* f) \) and hence the last integral vanishes. \( \square \)

On exact symplectic manifolds \((M, d\alpha)\), we can consider the action functional

\[
\mathcal{A}_H(\gamma) = - \int_{\gamma} \alpha - \int_0^1 H(t, \gamma(t)) dt
\]

and its first variation

\[
d\mathcal{A}_H(\gamma)(\xi) = \int_0^1 \omega(\xi, \dot{\gamma} - X_H(t, \gamma)) dt + \langle \alpha(\gamma(0)), \xi(\gamma(0)) \rangle - \langle \alpha(\gamma(1)), \xi(\gamma(1)) \rangle.
\]

(3.4.15)

This is nothing but (3.1.1) with \( \theta \) replaced by \(-\alpha\).

Again the periodic boundary condition will be a natural boundary condition that kills the boundary contribution.

**Question 3.4.5** Is there any other natural boundary condition for this functional \( \mathcal{A}_H \)?

**Remark 3.4.6** This kind of question is a prototype of the questions physicists often ask to correctly define the *equations of motion* for the given physical system with an appropriate *physical Lagrangian*. The objects that occur as the natural boundary conditions of the given system in *string theory* roughly correspond to the notion of *D-branes*, which has been playing a fundamental role in the current string theory since around 1996 when the physicists introduced the concept of *D-branes* in open string theory.
Now let us try to answer the above question. First we note that, when we impose boundary conditions
\[ \gamma(0) \in L_0, \quad \gamma(1) \in L_1 \]
for some submanifolds \( L_0, L_1 \), the boundary terms in (3.4.15) are nothing but
\[ i^*_0 \alpha(\xi(0)), i^*_1 \alpha(\xi(1)), \]
respectively. Now suppose that \( L_0, L_1 \) are exact Lagrangian submanifolds so that
\[ i^*_0 \alpha = df_0, \quad i^*_1 \alpha = df_1 \]
for some functions \( f_0 : L_0 \to \mathbb{R} \) and \( f_1 : L_1 \to \mathbb{R} \). In this case, we can modify our original action functional \( \mathcal{A}_H \) by adding the terms given by the functions \( f_0, f_1 \) and consider the modified action functional \( \mathcal{A}_{H; (f_0, f_1)} : \mathcal{P}(L_0, L_1) \to \mathbb{R} \) defined by
\[ \mathcal{A}_{H; (f_0, f_1)}(\gamma) = \mathcal{A}_H(\gamma) + f_1(\gamma(1)) - f_0(\gamma(0)). \]  
(3.4.16)
Then the first variation of this modified functional becomes
\[ d\mathcal{A}_{H; (f_0, f_1)}(\gamma)(\xi) = \int_0^1 \omega(\dot{\gamma} - X_H(t, \gamma(t)))dt \]
on \( \mathcal{P}(L_0, L_1) \) and hence the equation of motion of the modified action functional becomes
\[ \dot{z} = X_H(z), \quad z(0) \in L_0, \quad z(1) \in L_1, \]
which is exactly what we wanted. This motivates the following definition of exact Lagrangian branes specified by Seidel (Se03a), where some additional decorations are added to \( L \).

**Definition 3.4.7** (Exact Lagrangian branes) Consider \( (M, \omega) \) with \( \omega = d\alpha \). Let \( i : L \hookrightarrow (M, \omega) \) be an exact Lagrangian submanifold. We call the pair \( (L, f) \) satisfying \( df = i^* \alpha \) an exact Lagrangian brane.

Now we prove the following interesting formula on the exact Lagrangian branes by a computation similar to the case of a cotangent bundle.

**Proposition 3.4.8** Let \( (M, d\alpha) \) be an exact symplectic manifold, and let \( (L, g) \), an exact Lagrangian brane with \( i : L \subset M \), be the inclusion. Then the Hamiltonian flow \( \phi^t_H \) induces a smooth family of exact Lagrangian branes \( L_t, f_t \) provided by \( L_t = i_t(L), \quad i_t : L \to M \) and \( f_t : L \to \mathbb{R} \), where
3.5 Classical deformations of Lagrangian submanifolds

\[ i_t = \phi_H^t \circ i, \quad f_t = g + \int_0^t (H_s + \alpha(X_{H_s})) \circ i_s \, ds. \] (3.4.17)

**Proof** By definition of \( f_t \), we have \( g = f_0 \) and so \( i^* \alpha = df_0 \). We have to prove

\[ i_t^* \alpha = df_t \] (3.4.18)
for all \( t \in [0, 1] \). For this, we again compute the derivative of

\[ i_t^* \alpha = (\phi_H^t \circ i)^* \alpha = i^* (\phi_H^t)^* \alpha. \]

But simple calculation gives rise to

\[
\frac{d}{dt} (\phi_H^t)^* \alpha = (\phi_H^t)^* L_{X_H} \alpha \\
= (\phi_H^t)^* (X_{H_s}) \alpha + d \left( (\phi_H^t)^* (X_{H_s}) \alpha \right) \\
= d \left( H_t \circ \phi_H^t + \alpha(X_{H_s}) \circ \phi_H^t \right). \] (3.4.19)

Hence, on setting \( i_0^* \alpha = dg \), we obtain

\[
i_t^* \alpha = dg + \int_0^t \frac{d}{ds} i_s^* \alpha \, ds \\
= d \left( g + \int_0^t (H_s \circ \phi_H^s + \alpha(X_{H_s}) \circ \phi_H^s) \circ i_s \, ds \right) \\
= d \left( g + \int_0^t (H_s + \alpha(X_{H_s})) \circ i_s \, ds \right).
\]

Now this proves (3.4.18) with the definition of \( f_t \) given above. \qed

**Remark 3.4.9** We would like to point out that, even when \( L_1 = L_0 \), the function \( g_1 \) could be different from \( g_0 = g \). This monodromy phenomenon of the phase functions of exact Lagrangian branes gives rise to an interesting consequence in Seidel’s study of the Fukaya category of exact Lagrangian branes (Se03a). We refer the reader to (Se03a) for more discussion on this issue.

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3.5 Classical deformations of Lagrangian submanifolds

One immediate consequence of the Darboux–Weinstein theorem provides a complete description of local parameterization of nearby Lagrangian submanifolds of a given Lagrangian submanifold. We make this statement more precise now.

We introduce the following general definition.
Definition 3.5.1 Two Lagrangian submanifolds $L_0, L_1$ are Hamiltonian isotopic if there exists a time-dependent Hamiltonian $H : [0, 1] \times M \to \mathbb{R}$ such that

$$L_1 = \phi^1_H(L_0).$$

We call the isotopy $\{L_t = \phi^t_H(L_0)\}$ a Hamiltonian isotopy of $L_0$.

3.5.1 The case of the zero section

We recall that the graph of a one-form, or a section of the cotangent bundle $T^*M \to M$, is Lagrangian if and only if the one-form is closed. Furthermore, we know that any Lagrangian submanifold sufficiently $C^1$-close to the zero section $o_L \subset T^*L$ is the graph of a one-form. In particular, if we regard the set $\text{Lag}(T^*L)$ of Lagrangian submanifolds of $T^*L$ as an infinite-dimensional (Fréchet) manifold, then the neighborhood of the zero section is modeled by $Z^1(L)$, the set of closed one-forms. Furthermore, any exact Lagrangian submanifold $C^1$-close to the zero section is Hamiltonian isotopic to the zero section. This is because any such Lagrangian submanifold has the form $\text{Graph}(df)$ of some function $f$ that is $C^2$-close to the constant function and then the isotopy

$$t \in [0, 1] \to \text{Graph}(t df)$$

provides such a Hamiltonian isotopy.

Exercise 3.5.2 Prove that the above isotopy of the zero section is induced by the (autonomous) function $H = f \circ \pi : T^*L \to \mathbb{R}$.

Exercise 3.5.3 Prove that there exists a contractible $C^\infty$ neighborhood of the identity map in $\text{Symp}(M, \omega)$ (respectively $\text{Ham}(M, \omega)$). (Hint. Consider the correspondence between a symplectic diffeomorphism $\phi : M \to M$ and its graph $\text{Graph}(\phi \subset M \times M$ as a Lagrangian submanifold with respect to the form $(-\omega) \oplus \omega$ on $M \times M$.)

More generally, we have the following local description of Lagrangian submanifolds up to the Hamiltonian isotopy on the cotangent bundle.

Proposition 3.5.4 The set of Lagrangian submanifolds $C^1$ close to the zero section modulo the $C^1$-small Hamiltonian isotopy is diffeomorphic to an open neighborhood of the zero in $H^1(L, \mathbb{R})$. 

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**Proof** Let $\mathcal{N}$ be a $C^1$-neighborhood in $\text{Lag}(T^*L)$ of the zero section. As we mentioned above, we can write

$$\mathcal{N} = \{ \text{Graph} \alpha \mid \alpha \in Z^1(L), |\alpha|_{C^1} < \delta \} \quad (3.5.20)$$

for some $\delta > 0$. Denote by $L_\alpha$ the Lagrangian submanifold corresponding to $\alpha$. We consider the map $L_\alpha \in \mathcal{N} \to [\alpha] \in H^1(L, \mathbb{R})$.

Obviously this map is surjective to an open neighborhood of zero in $H^1(L, \mathbb{R})$. Suppose that $L_\alpha$ and $L_{\alpha'}$ are Hamiltonian isotopic, i.e., $L_{\alpha'} = \phi_H^t(L_\alpha)$ for a function $H : [0, 1] \times M \to \mathbb{R}$ such that $\phi_H^t(L_\alpha) \subset \mathcal{N}$ for all $t \in [0, 1]$. Then we can write

$$\phi_H^t(L_\alpha) = L_{\alpha_t} \quad (3.5.21)$$

for a family of closed one-forms $\alpha_t$. To compute the cohomology class $[\alpha_t]$, we evaluate $\int_c \alpha_t$ for a loop $c : S^1 \to L$. Identifying the one-forms $\alpha$, $\alpha_t$ with the maps $\tilde{\alpha}$, $\tilde{\alpha_t} : L \to T^*L$, we have

$$\tilde{\alpha_t} = \phi_H^t \circ \tilde{\alpha}.$$ 

Therefore, recalling $\beta = (\tilde{\beta})^* \theta$, we have

$$\int_c \alpha_t = \int_c (\alpha_t)^*(\theta) = \int_c (\tilde{\alpha})^* (\phi_H^t)^* \theta = \int_{\tilde{\alpha} \circ c} (\phi_H^t)^* \theta.$$ 

We compute

$$\int_c \alpha' - \int_c \alpha = \int_0^1 \frac{d}{dt} \left( \int_c \alpha_t \right) dt = \int_0^1 \frac{d}{dt} \left( \int_{\tilde{\alpha} \circ c} (\phi_H^t)^* \theta \right) dt$$

$$= \int_0^1 \left( \int_{\tilde{\alpha} \circ c} \frac{d}{dt} (\phi_H^t)^* \theta \right) dt.$$ 

Substituting (3.4.19) for $\alpha = -\theta$ into the above integral, we derive

$$\int_{\tilde{\alpha} \circ c} \frac{d}{dt} (\phi_H^t)^* \theta = 0$$

and hence $\int_c \alpha' - \int_c \alpha = 0$ for any loop $c$. This proves $[\alpha] = [\alpha']$ in $H^1(L; \mathbb{R})$ and so the assignment induces a well-defined surjective map

$$\mathcal{N}/\text{Ham} \to H^1(L, \mathbb{R})$$

onto an open neighborhood of zero in $H^1(L, \mathbb{R})$. It remains to prove its injectivity. Now suppose $[\alpha] = [\alpha']$, i.e., $\alpha' - \alpha = df$. The injectivity follows from Exercise 3.5.5 below similarly to Exercise 3.5.2. This finishes the proof. \qed
Exercise 3.5.5  Prove that the map
\[ t \mapsto \alpha_t := \alpha + tdf, \]
and hence its associated Lagrangian submanifold \( L_{\alpha_t} \), is induced by the Hamiltonian isotopy associated with it by the function \( H = f \circ \pi \).

3.5.2 The general case

Now we consider a general \((M, \omega)\). Let \( L \subset M \) be a Lagrangian embedding. The Darboux–Weinstein theorem provides a diffeomorphism
\[ \Phi : U \subset M \to V \subset T^*L \]
extending the map \( L \to o_L \) such that \( \Phi^*\omega_0 = \omega \). We call \( \Phi \) the cotangent bundle chart of \( L \).

Note that all the Lagrangian submanifolds sufficiently \( C^1 \)-close (indeed \( C^0 \)-close suffices for this purpose) to the given Lagrangian submanifold \( L \subset M \) are contained in a given Darboux chart \( \Phi \) mapping \( L \) to \( o_L \subset T^*L \). From the discussion on the cotangent bundle above, it follows that we can parameterize any such Lagrangian submanifold \( C^1 \)-close to \( L \subset M \) by closed one-forms on \( L \) contained in an open \( C^1 \)-neighborhood of the zero in the set \( Z^1(L) \). We would like to emphasize that this parameterization depends on the cotangent bundle chart, but its infinitesimal version does not.

We first study the transition map of two cotangent bundle charts \( \Phi_1, \Phi_2 \) extending the canonical identification \( Z : L \to o_L \subset T^*L \),
\[ \Phi_{12} = \Phi_2 \circ (\Phi_1)^{-1} : V \to V'. \]
It satisfies \( \Phi_{12}^*\omega_0 = \omega_0 \) and \( \Phi_{12}|_{o_L} = id|_{o_L} \). Recall that the tangent space \( T_L \text{Sub}(M) \) is canonically isomorphic to \( \Gamma(\nu(L)) \), where \( \nu(L) = TM/TL \) is the normal bundle of \( L \) and \( \text{Sub}(M) \) is the set of \( n \) dimensional submanifolds of \( M \).

Lemma 3.5.6  We have the canonical isomorphism \( \Gamma(\nu(o_L)) \cong \Gamma(T^*L) \) obtained by \( [X] \mapsto (X|\omega_0)|_{T_{o_L}} \), where \( X \) is a vector field along \( o_L \). Under this identification we have the natural identification
\[ T_{o_L} \text{Lag}(T^*L) \cong Z^1(L), \]  
(3.5.22)
under which \( d\Phi_{12} \) induces the identity map on \( \Gamma(T^*L) \).

Proof  For the first statement, we recall the canonical splitting \( T(T^*L)|_{o_L} = TL \oplus T^*L \). First we prove the well-definedness of the above map. Let \( X, X' \) be
two vector fields along $o_L$ such that $[X] = [X']$ on $\Gamma(\nu(o_L))$. For simplicity, we will just denote $o_L = L$. Then we have $X(x) - X'(x) \in T_xL$ for all $x \in L$. Since $L$ is Lagrangian, we have

$$(X - X')(x)\omega_0|_{T_xL} = 0,$$

i.e., $X(x)\omega_0|_{T_xL} = X'(x)\omega_0|_{T_xL}$

for all $x \in L$. Hence we have $[X]\omega_0 = [X']\omega_0$ and thus the map is well defined. Similarly, we have that the map is bijective, the proof of which we leave to the reader.

We have already shown that the tangent vector of $\text{Sub}(T^*L)$ corresponding to the one-form $\alpha \in \Gamma(T^*L)$ is tangent to $\text{Lag}(T^*L)$ at $o_L$ if and only if $\alpha$ is closed. This establishes the isomorphism (3.5.22).

For a given one-form $\alpha$ on $L$, we denote by $X_\alpha$ the associated vector field along $o_L$ defined by the equation $X_\alpha\omega_0|_{TL} = \alpha$. Then the variation $d\Phi_{12}(X_\alpha)$ corresponds to the element of $\Gamma(T^*L)$ which is defined by $d\Phi_{12}(X_\alpha)|_{T_L}$ under the isomorphism (3.5.22).

To evaluate this section in $\Gamma(T^*L)$, we explicitly compute

$$d\Phi_{12}(X_\alpha)|_{T_L}(Y) = \omega_0(d\Phi_{12}(X_\alpha)Y)$$

$$= \omega_0(X_\alpha, (d\Phi_{12})^{-1}Y)$$

$$= \omega_0(X_\alpha, Y) = \alpha(Y)$$

for each given vector field $Y \in T_L$, which finishes the proof. Here we have used the fact that $\Phi_{12}$ is symplectic for the second identity, $\Phi_{12}|_{o_L} = id|_{o_L}$ for the third and the definition of $X_\alpha$ for the last identity. This finishes the proof. 

We use this lemma to prove the following canonical representation of the tangent space $T_{L,\text{Lag}}(M, \omega)$.

**Proposition 3.5.7** Let $\text{Lag}_L(M, \omega)$ be the set of Lagrangian submanifolds of its topological type $L$. Then there exists a canonical isomorphism

$$T_{L, \text{Lag}}(M, \omega) \cong Z^1(L).$$

**Proof** Since $L$ is Lagrangian, the symplectic form $\omega$ gives the natural isomorphism

$$\nu(L) \to (TL)^* = T^*L; [v] \mapsto \nu(v)|_{TL}. $$

Therefore we have a natural isomorphism $T_{L, \text{Sub}}(M) \cong \Gamma(T^*L)$. On the other hand, for any given cotangent bundle chart, $\text{Lag}_L(M, \omega)$ near $L$ is parameterized by the set of $C^1$-small closed one-forms on $L$ and hence its tangent space
is also parameterized by $Z^1(L)$. Lemma 3.5.6 then implies that the parameterization of the tangent space does not depend on the choice of cotangent bundle charts. This finishes the proof. □

The tangent bundle $T\text{Lag}(M,\omega) \to \text{Lag}(M,\omega)$ contains a canonical distribution spanned by the set of exact one-forms $B^1(L) \subset Z^1(L) \cong T_L\text{Lag}(M,\omega)$ at $L \in \text{Lag}(M,\omega)$. We denote by $E \to \text{Lag}(M,\omega)$ the corresponding subbundle. Then this defines the distribution of corank $b^1(L) = \dim H^1(L,\mathbb{R})$.

We will give the proof of the following theorem stated in (Wn90) in Section 3.6.

**Theorem 3.5.8** The distribution $E \subset T\text{Lag}(M,\omega)$ is an integrable sub-bundle whose leaf $L_L \subset \text{Lag}(M,\omega)$ passing through a given Lagrangian submanifold $L \in \text{Lag}(M,\omega)$ consists of the Hamiltonian deformations of $L$. More precisely, for any $L' \in L_L$ there exists a Hamiltonian diffeomorphism $\phi \in \text{Ham}(M,\omega)$ such that $L' = \phi(L)$. In fact, $\text{Ham}(M,\omega)$ acts on $\text{Lag}(M,\omega)$ and $E_L$ is the tangent space of the orbit of $\text{Ham}(M,\omega)$ on $L$.

### 3.6 Exact Lagrangian isotopy = Hamiltonian isotopy

In this section, following Gromov (Gr85), we will introduce the notion of exact Lagrangian isotopy on arbitrary $(M,\omega)$ as a particular type of smooth path in $\text{Lag}(M,\omega)$ and prove that the notion is equivalent to that of isotopy of (embedded) Lagrangian submanifolds obtained by the ambient Hamiltonian flow on the ambient manifold $(M,\omega)$. We call such an isotopy an ambient Hamiltonian isotopy. Such an extension property fails for the more general Lagrangian isotopy, which is one of the essential differences between the two types of isotopy.

As a corollary, we will give the proof of Theorem 3.5.8 at the end.

Let $\psi : [0,1] \times L \to M$ be an isotopy of embeddings $\psi_t : L \to M$. The pull-back $\psi^*\omega$ is decomposed to

$$
\psi^*\omega = dt \wedge \alpha + \beta,
$$

where $\alpha$ and $\beta$ are forms on $L$ such that

$$
\frac{\partial}{\partial t}|\alpha = 0 = \frac{\partial}{\partial t}|\beta.
$$

In fact, we can write

$$
\alpha = \frac{\partial}{\partial t}|\psi^*\omega, \quad \beta = \psi^*\omega - dt \wedge \alpha
$$
purely in terms of $\psi^\ast \omega$. One may regard $\alpha, \beta$ as a $t$-dependent family of forms on $L$.

**Definition 3.6.1** An isotopy $\psi : [0,1] \times L \to M$ is called a *Lagrangian isotopy* if $d\beta = 0$ in the above decomposition.

**Proposition 3.6.2** Denote by $i_t : L \to [0,1] \times L$ the obvious inclusion defined by $i_t(x) = (t,x)$. Then an isotopy $\psi$ is Lagrangian if and only if $i_t^\ast \alpha$ is a closed one-form on $L$ for all $t \in [0,1]$.

*Proof* Suppose that $d\beta = 0$. Then, since $\omega$ is closed, we have $0 = dt \wedge d\alpha$ by taking the exterior derivative of (3.6.23). By taking the interior product with $\partial/\partial t$, we derive

$$d\alpha - dt \wedge \left( \frac{\partial}{\partial t} d\alpha \right) = 0.$$ 

By pulling back this equation by $i_t$, we obtain $d(i_t^\ast \alpha) = 0$.

Conversely, suppose that $d(i_t^\ast \alpha) = 0$. Then we have $i_t^\ast (d\alpha) = 0$. In other words, $d\alpha(X, Y) = 0$ whenever $X, Y$ are tangent to the fibers of $[0,1] \times L \to [0,1]$. Therefore the three-form $dt \wedge d\alpha$ should vanish on $[0,1] \times L$. Again taking the exterior derivative of (3.6.23) and using the closedness of $\psi^\ast \omega$, we obtain $d\beta = 0$, which finishes the proof. 

This leads us to the following definition.

**Definition 3.6.3** A Lagrangian isotopy $\psi$ is *exact* if $i_t^\ast \alpha$ is exact for all $t \in [0,1]$. We call such an isotopy an *exact Lagrangian isotopy*.

**Lemma 3.6.4** If $\psi$ is an exact Lagrangian isotopy, then there exists a function $h : [0,1] \times L \to \mathbb{R}$ such that $\psi^\ast \omega = dh \wedge dt$. Furthermore, if $h'$ is another such function, $h' - h$ is a function depending only on $t$, but not on $L$.

*Proof* By definition, we know that there exists $h_t : L \to \mathbb{R}$ such that $i_t^\ast \alpha = dh_t$ for each $t \in [0,1]$. This choice is unique up to addition of a constant $c(t)$. Here we recall that, by the remark right before Definition 3.6.1, the forms $\alpha$ and $\beta$ are uniquely determined by $\psi$. This proves the last statement of the lemma. The only thing we need to ensure is the smooth dependence of $h$ on $t$. This can be achieved by requiring $h_t(x_0) = 0$ for all $t \in [0,1]$ at a fixed point $x_0$ taken from each connected component of $L$. Then we define $h : [0,1] \times L \to \mathbb{R}$ by $h(t,x) = h_t(x)$.
Exercise 3.6.5  Complete the above proof. More precisely, prove that the above function $h$ is indeed smooth and satisfies $\psi^* \omega = dh \wedge dt$.

Remark 3.6.6  We would like to point out that the above definitions of Lagrangian isotopy and exact Lagrangian isotopy can be also made for immersions, not just for embeddings.

The following extension theorem distinguishes the exact Lagrangian isotopy of embeddings either from general Lagrangian isotopies or from exact Lagrangian isotopies of immersions. Because of this theorem, we often do not distinguish between the two terminologies, the ‘Hamiltonian isotopy’ and the ‘exact Lagrangian isotopy’, for the embeddings.

Theorem 3.6.7  For any exact Lagrangian isotopy $\psi : [0, 1] \times L \to M$ of Lagrangian embeddings, there exists $H : [0, 1] \times M \to \mathbb{R}$ such that
\[
\psi(t, x) = \phi_t^H(\psi(0, x)) = \phi_t^H(\psi_0(x)).
\] (3.6.24)

We will prove the theorem in two steps. Firstly, we find a function $H' : [0, 1] \times M \to \mathbb{R}$ such that
\[
\frac{\partial \psi}{\partial t} \equiv X_{H'}(\psi(t, x)) \big|_{L_t} \operatorname{mod} TL_t, \quad (3.6.25)
\]
where $L_t = \psi_t(L)$ is the image of $\psi_t$. Secondly, we will modify $H'$ to $H$ so that
\[
\frac{\partial \psi}{\partial t} = X_H(\psi(t, x)) \quad (3.6.26)
\]
for $(t, x) \in [0, 1] \times L$. We start with the following general lemma.

Lemma 3.6.8  Let $L$ be an $n$-dimensional manifold.

1. If $g : L \to (M, \omega)$ is a Lagrangian embedding, and $\xi \in \Gamma(TL)$ is a vector field on $L$, there exists a function $f : M \to \mathbb{R}$ such that
\[
X_f(g(x)) = dg(\xi(x)), \quad x \in L.
\] (3.6.27)

2. For a given Lagrangian isotopy $\psi : [0, 1] \times L \to M$ of embeddings and $\xi = \{\xi_t\}_{0 \leq t \leq 1}$ a smooth family of vector fields on $L$, there exists a smooth function $F : [0, 1] \times M \to \mathbb{R}$ such that
\[
X_{F_t}(\psi_t(x)) = d\psi_t(\xi_t(x))
\] (3.6.28)
for all $t \in [0, 1]$. 

Proof  The Darboux–Weinstein theorem reduces the proof of statement (1) to the case $L = o_L$ and $M = T^*L$ with the canonical symplectic form $\omega_0$ by multiplying a cut-off function. But, if $\xi$ is a vector field on $o_L$, we can explicitly choose the function $f_\xi : T^*L \to \mathbb{R}$ defined by

$$f_\xi(\alpha) = \langle \alpha, \xi(\pi(\alpha)) \rangle.$$  \hfill (3.6.29)

The following exercise will finish the proof of this claim.

Exercise 3.6.9  Prove $X_{f_\xi} \circ g = \xi$ on $L$.

For statement (2), we use the parameterized Darboux–Weinstein theorem (Exercise 3.3.4) and define $F_\xi : [0, 1] \times T^*L \to \mathbb{R}$ by

$$F(t, \alpha) = \langle \alpha, \xi_t(\pi(\alpha)) \rangle,$$

which finishes the proof. $\square$

Another important lemma is the following general extension lemma.

Lemma 3.6.10 (Extension lemma)  Let $\pi : E \to N$ be a vector bundle and $h : N \to \mathbb{R}$ be a $C^1$-function. Then there exists a $C^1$-function $H' : E \to \mathbb{R}$ such that

$$H'|_N = h, \quad dH'|_{TN} = d(h \circ \pi)|_{TN},$$

and the support of $H'$ can be put into $U$, a neighborhood of $N$ that can be made as small as we want.

Proof  Let $r_t : E \to E$ be the retraction of $E$ onto the zero section $o_E \cong N$, which is defined by

$$r_t(e) = \begin{cases} te, & 0 \leq t \leq 1, \\ \pi(e), & t = 0 \end{cases}$$

for $e \in E$. We fix a map $\chi : [0, 1] \to [0, 1]$ such that $\chi(t) \equiv 0$ near $t = 0$, $\chi(1) = 1$ and $\chi'(t) \geq 0$.

If we define a time-dependent vector field $X_t$ by

$$X_t = \begin{cases} d\chi(t)/dt \circ r^{-1}_{\chi(t)}, & 0 < t \leq 1, \\ 0, & t = 0 \end{cases}$$

we have

$$\frac{d}{dt} r_{\chi(t)}^* \alpha = r_{\chi(t)}^*(d(X_t|\alpha) + X_t|d\alpha)$$
for $0 \leq t \leq 1$. When a form $\alpha$ is closed, we have
\[
\frac{d}{dt} r_{\chi(t)}^* \alpha = r_{\chi(t)}^* (d(X_t \lceil \alpha)).
\]
Therefore we derive
\[
\alpha - r_0^* \alpha = \int_0^1 r_{\chi(t)}^* (d(X_t \lceil \alpha)) dt = \int_0^1 r_{\chi(t)}^* (X_t \lceil \alpha) dt,
\]
which implies
\[
\alpha = \pi^* \alpha + d \int_0^1 r_{\chi(t)}^* (X_t \lceil \alpha) dt
\]
for a closed one-form $\alpha$. We apply this to the form $\alpha = d(\rho \circ \pi), h \circ \pi : E \to \mathbb{R}$, where $\rho : E \to \mathbb{R}$ such that
\[
\rho = \begin{cases} 1, & \text{on a neighborhood } U \text{ of } N, \\ 0, & \text{away from } N. \end{cases}
\]
Now we consider the function $H' : E \to \mathbb{R}$ given by
\[
H' = \rho (h \circ \pi) + \int_0^1 r_{\chi(t)}^* (X_t \lceil \rho \circ \pi) dt.
\]
Since $r_{\chi(t)}|_N = id|_N$ and $X_t \equiv 0$ on $N$ for all $t \in [0, 1]$, we have $H'|_N = h$ and $dH'|_{T_N} = d(h \circ \pi)|_{T_N}$. This finishes the proof. \qed

**Exercise 3.6.11** Formulate and prove the parameterized version of the extension lemma.

Using these lemmata, we give the proof of Theorem 3.6.7.

**Proof of Theorem 3.6.7** Note that $\partial \psi / \partial t$ is a vector field along $\psi$, i.e., $(\partial \psi / \partial t)(x) \in T_{\psi(x)} M$. By Lemma 3.5.6, (3.6.25) is equivalent to
\[
\frac{\partial \psi}{\partial t} \lceil \omega = dH'(\psi(t, x))|_{L_t} \in \Gamma(T^* L_t) \quad (3.6.30)
\]
for $t \in [0, 1]$. On the other hand, since $\psi$ is exact Lagrangian isotopy, we have
\[
\psi^* \omega = dh' \wedge dt
\]
for some $h' : [0, 1] \times L \to \mathbb{R}$, i.e.,
\[
dh'_t = \frac{\partial}{\partial t} |_{\psi^* \omega} = \omega \left( d\psi_t \left( \frac{\partial}{\partial t} \right), d\psi_t(\cdot) \right).
\]
Therefore (3.6.30) is reduced to
\[ d(h'_t \circ \psi_t^{-1})|_{TL_t} = \frac{\partial \psi}{\partial t}\bigg|_{\omega}. \]
(Here we use the embeddedness of the maps \( \psi_t \) in an essential way.) Setting \( h(t, y) = h'_t \circ \psi_t^{-1}(y) \) for \( y \in \psi_t(L) \), (3.6.30) is reduced to the parameterized extension lemma in the above exercise. Therefore we have finished the first step of the proof, finding \( H' \) on \( M \) that satisfies (3.6.25).

Then the vector field
\[ \frac{\partial \psi}{\partial t}(x) - X_{H'}(t, \psi(t, x)) \]
is tangent to \( \psi_t(L) \). Hence it follows from Lemma 3.6.8 that we can find \( F : [0, 1] \times M \to \mathbb{R} \) such that
\[ \frac{\partial \psi}{\partial t}(x) - X_{H'}(t, \psi(t, x)) = X_F(t, \psi(t, x)) \]
for \((t, x) \in [0, 1] \times L\). Then we choose \( H = H' + F \), which finishes the proof of the theorem.

Now we are ready to prove Weinstein’s theorem, Theorem 3.5.8. By Theorem 3.6.7, it suffices to prove that, at any \( L \in \text{Lag}(M, \omega) \),
\[ B^1(L) = \text{span}[X_f|_{\omega}|_{TL} \mid f : M \to \mathbb{R}] = T_L(\text{Ham}(M, \omega) \cdot L). \]
However, this is an immediate consequence of Theorem 3.6.7. Hence the proof.

---

3.7 Construction of Lagrangian submanifolds

3.7.1 Lagrangian surgery

Let \( L_1, L_2 \) be a pair of oriented Lagrangian submanifolds in \((M, \omega)\) that intersect transversely at \( p_{12} \). We fix an ordering of the pair as \((L_1, L_2)\). We can always choose a Darboux chart in a neighborhood \( U \) of \( p_{12} \), \( I : U \to V \subset \mathbb{C}^n \) so that \( I(p_{12}) = 0 \),
\[ I(L_1 \cap U) = \mathbb{R}^n \cap V, \quad I(L_2 \cap U) = \sqrt{-1} \mathbb{R}^n \cap V. \]
We explain this construction in a way that is more detailed than necessary, hoping that this concrete description may be useful for the more geometric study of Lagrangian surgery in relation to the study of metamorphosis of the moduli space of holomorphic polygons under the surgery. See (FOOO07) for such an application.

The proof follows from a version of Darboux’ theorem but strongly relies on the following well-known fact in symplectic linear algebra.
Lemma 3.7.1  The linear symplectic group $Sp(2n)$ acts transitively on the set of transversal pairs of Lagrangian subspaces.

Exercise 3.7.2  Prove this lemma.

We would like to point out that $U(n) \subset Sp(2n)$ does not act transitively on the set of such pairs. For example, the action of the unitary group $U(n)$ preserves the Hermitian structure of $\mathbb{C}^n$ and so preserves the ‘angles’ between the two Lagrangian subspaces.

Let $\epsilon$ be a real number sufficiently close to 0. We choose the function $f_\epsilon : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ defined by

$$f_\epsilon(x) = \epsilon \log|x| \quad (3.7.31)$$

and denote the graph $\text{Graph} \, df_\epsilon$ by $H_\epsilon \subset \mathbb{C}^n$. This is a Lagrangian submanifold in $T^* (\mathbb{R}^n \setminus \{0\}) \cong \mathbb{C}^n \setminus \sqrt{-1} \mathbb{R}^n$ that is asymptotic to $\sqrt{-1} \mathbb{R}^n$ as $|x| \to 0$, and to $\mathbb{R}^n$ as $|x| \to \infty$. Noting that we have

$$df_\epsilon(x) = \epsilon \frac{x \cdot dx}{|x|^2} = \frac{\epsilon}{|x|^2} \sum_{j=1}^{n} x_j \, dx_j,$$

we can write

$$H_\epsilon = \left\{ (z_1, \ldots, z_n) \mid y_j = \frac{\epsilon x_j}{|x|^2}, \, j = 1, \ldots, n \right\} \quad (3.7.32)$$

in coordinates. Here we denote the complex coordinates of $\mathbb{C}^n$ as $z_j = x_j + \sqrt{-1} y_j$ for $j = 1, \ldots, n$.

Let $\tau : \mathbb{C}^n \to \mathbb{C}^n$ be the reflection along the diagonal $\Delta = \{(z_1, z_2, \ldots, z_n) \mid x_i = y_i\}$, i.e., be the map

$$(x_1 + \sqrt{-1} y_1, \ldots, x_n + \sqrt{-1} y_n) \mapsto (y_1 + \sqrt{-1} x_1, \ldots, y_n + \sqrt{-1} x_n).$$

We remark that (3.7.32) implies $|x|^2 |y|^2 = |\epsilon|^2$, and hence

$$H_\epsilon = \left\{ (z_1, \ldots, z_n) \mid x_j = \frac{\epsilon y_j}{|y|^2}, \, j = 1, \ldots, n \right\}.$$  

In other words $\tau(H_\epsilon) = H_\epsilon$. It is easy to check that

$$\inf\{|\tilde{z}| \mid \tilde{z} \in H_\epsilon\} = \sqrt{2|\epsilon|},$$

which is achieved at the points on the $(n-1)$-sphere

$$\{(x, y) \mid |x| = |y| = \sqrt{\epsilon} \} \subset H_\epsilon.$$
Next we consider a function $\rho : \mathbb{R}_+ \to \mathbb{R}$ such that
\[
\rho = \begin{cases} 
\log r - |c|, & \text{if } r \leq \sqrt{|c|} S_0, \\
\log \sqrt{|c|} S_0, & \text{if } r \geq 2 \sqrt{|c|} S_0,
\end{cases}
\]
where $S_0$ is a sufficiently large number and $c$ is chosen so that $\sqrt{|c|} S_0$ is sufficiently small. We then define the function \( \tilde{f}_\epsilon : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) by
\[
\tilde{f}_\epsilon(x) = \epsilon \rho(|x|).
\]
(3.7.33)

Consider the graph $\text{Graph } d\tilde{f}_\epsilon$ and define a Lagrangian submanifold $H'_\epsilon$ so that the following hold.

1. \( \tau(H'_\epsilon) = H'_\epsilon \).
2. \[ \{ (x_1 + \sqrt{-1} y_1, \ldots, x_n + \sqrt{-1} y_n) | \forall i \ x_i \geq y_i \} \cap H'_\epsilon = \{ (x_1 + \sqrt{-1} y_1, \ldots, x_n + \sqrt{-1} y_n) | \forall i \ x_i \geq y_i \} \cap \text{Graph } d\tilde{f}_\epsilon. \]

By construction, $H'_\epsilon$ is invariant under $\tau$ and $H'_\epsilon = \mathbb{R}^n \cup \sqrt{-1} \mathbb{R}^n$ outside the ball $B^{2n}(2 \sqrt{|c|} S_0)$ around 0 in $I(U) \subset \mathbb{R}^{2n}$. Therefore, for a given ordered pair $(L_1, L_2)$, we can construct a Lagrangian submanifold $L_\epsilon \subset M$ such that
\[
L_\epsilon - U = L_1 \cup L_2 - U, \quad I(L_\epsilon \cap U) = H'_\epsilon \cap V.
\]

**Definition 3.7.3** For a given ordered pair $(L_1, L_2)$, we call $L_\epsilon$ the Lagrangian connected sum of $L_1$ and $L_2$ at $p_{12} \in L_1 \cap L_2$ and write $L_1 #_\epsilon L_2 = L_\epsilon$.

Note that, if we change the ordering of the pair $(L_1, L_2)$ at $p \in L_1 \cap L_2$ and change the sign of $\epsilon$ at the same time, then the resulting Lagrangian submanifolds are isomorphic. In fact, we have
\[
\mathbb{R}^n #_\epsilon \sqrt{-1} \mathbb{R}^n = \sqrt{-1} \mathbb{R}^n #_{-\epsilon} \mathbb{R}^n.
\]

We call the pre-image
\[
(L_1 #_\epsilon L_2) \cap U = I^{-1}(H'_\epsilon \cap V)
\]
a Lagrangian handle and its meridian sphere $S^{n-1}$ a vanishing cycle of the Lagrangian surgery $L_\epsilon$.

We say that the pair $L_1, L_2$ or its associated Lagrangian surgery $L_1 #_\epsilon L_2$ with $\epsilon > 0$ is positive at $p_{12}$ if
\[
T_{p_{12}} L_1 \oplus T_{p_{12}} L_2 = (-1)^{(n-1)/2+1} T_{p_{12}} M
\]
as an oriented vector space and negative otherwise. (Here we equip $T_{p_1}M$ with the symplectic orientation.) For example, for $L_1 = \mathbb{R}$, $L_2 = \sqrt{-1}\mathbb{R} \subset \mathbb{C}$ with the standard orientation on $\mathbb{R}$, $\sqrt{-1}\mathbb{R}$, the Lagrangian surgery $L_1#_{\epsilon}L_2$, $\epsilon > 0$, is negative. (This example can be directly extended to the case of $L_1 = \mathbb{R}^n$, $L_2 = \sqrt{-1}\mathbb{R}^n \subset \mathbb{C}^n$.) It is easy to check that only the positive surgery allows one to glue the orientations on $L_1$ and $L_2$ to have the surgery $L_1#_{\epsilon}L_2$ carry a compatible orientation. (In the case of $L_1 = \mathbb{R}$, $L_2 = \sqrt{-1}\mathbb{R} \subset \mathbb{C}$, it is easy to see that it is impossible to give an orientation of $L_1#_{\epsilon}L_2$ that is compatible with both standard orientations of $L_i$. A similar remark also holds for $L_1 = \mathbb{R}^n$, $L_2 = \sqrt{-1}\mathbb{R}^n \subset \mathbb{C}^n$.)

We remark that $L_1#_{\epsilon}L_2$ is not even isotopic to $L_2#_{\epsilon}L_1$ (or $L_1#_{-\epsilon}L_2$) in general, even when both are orientable. On the other hand, it is easy to check that $L_1#_{\epsilon}L_2$ are Lagrangian isotopic to one another among different $\epsilon$ of the same sign. However, they are not Hamiltonian isotopic to one another with a fixed boundary in general.

Since the above construction is local in a neighborhood of the intersection point, one can perform the surgery at a self-intersection point of a Lagrangian immersion. Let $L \subset M$ be an immersed Lagrangian submanifold with a transverse self-intersection $p \in L \subset M$. Consider a small neighborhood $U$ of $p$ in $M$ and the intersection $U \cap L$. There are two branches in $U \cap L$ at $p$. To carry out the Lagrangian surgery, we need to order the two branches. Following the terminology from (Po91b), we introduce the following definition.

**Definition 3.7.4** Let $L$ and $p$ be as above. An equipment at $p$ is an ordering of the two branches of $L$ at $p$, or equivalently an ordering of the corresponding tangent spaces at $p$.

For a choice of equipment at a self-intersection point of $L$, we can perform Lagrangian surgery and decrease the number of intersection points. In this way, one can obtain a Lagrangian embedding out of a Lagrangian immersion. By Gromov’s h-principle, the obstruction to the existence of Lagrangian immersions is purely topological, while there exist subtler hard obstructions to the existence of Lagrangian embedding (see (Gr85) for such theorems). Unfortunately, the embeddings obtained from Lagrangian surgery tend to have complicated topology.

**Example 3.7.5** Gromov (Gr85) proved that any compact embedded Lagrangian submanifold $L$ in $\mathbb{C}^n$ should satisfy $H_1(L; \mathbb{R}) \neq 0$. In particular, there does not exist an embedded Lagrangian sphere $S^n$ in $\mathbb{C}^n$. But there is a famous immersed sphere in $\mathbb{C}^n$, called the Whitney sphere, which can be made
Lagrangian. In fact we have the following explicit formula for the immersion \( \psi : S^n \to \mathbb{C}^n \) given by

\[
\psi(x_1, \ldots, x_{n+1}) = \frac{1}{1 + x_{n+1}^2} (x_1(1 + ix_{n+1}), x_2(1 + ix_{n+1}), \ldots, x_n(1 + ix_{n+1})).
\]  

(3.7.34)

This has one transverse self-intersection point at the origin 0 of \( \mathbb{C}^n \) whose pre-images are the north and south poles of \( S^n \subset \mathbb{R}^{n+1} \). One can easily check that \( \psi \) is also Lagrangian. By performing the Lagrangian surgery of this at the intersection, we will get a Lagrangian embedding of \( S^1 \times S^{n-1} \) into \( \mathbb{C}^n \).

The case \( n = 2 \) is particularly interesting. We denote by \((z_1, z_2)\) the complex coordinates of \( \mathbb{C}^2 \). In this case, the formula is reduced to

\[
\psi(x, y, z) = \frac{1}{1 + z^2} (x(1 + iz), y(1 + iz)), x^2 + y^2 + z^2 = 1,
\]

(3.7.35)

for \( x, y, z \) real numbers.

The two branches of the tangent space at 0 are given by

\[
V_+ = \text{span}_\mathbb{R} \left\{ (1 + i) \frac{\partial}{\partial z_1}, (1 + i) \frac{\partial}{\partial z_2} \right\}, \quad \text{for } z = 1,
\]

\[
V_- = \text{span}_\mathbb{R} \left\{ (1 - i) \frac{\partial}{\partial z_1}, (1 - i) \frac{\partial}{\partial z_2} \right\}, \quad \text{for } z = -1.
\]

One can easily see that \( V_+ + V_- = \mathbb{C}^2 \). Denote by \( e_1, e_2 \) the equipments given by the ordering

\[
e_1 = \{ V_+, V_- \}, \quad e_2 = \{ V_-, V_+ \},
\]

and denote the associated Lagrangian surgery by

\[
L_\epsilon^e, \quad \epsilon > 0.
\]

Both have the topological type of the torus \( T^2 \). It turns out that \( L_\epsilon^{e_1} \) is Hamiltonian isotopic to the standard product torus \( S^1 \times S^1 \subset \mathbb{C}^2 \) for a suitable choice of \( \epsilon_1 \) but \( L_\epsilon^{e_2} \) is not. The latter torus has been shown to be isomorphic to the Chekanov torus \( \text{(Che96b), (EP97)} \), whose definition will be explained later.

### 3.7.2 Lagrangian suspension

In this section, we denote a general symplectic manifold by \((X, \omega)\), instead of using the letter \( M \). We also denote a closed manifold by \( Y \). When we are given a Lagrangian embedding of \( Y \) into \((X, \omega)\), we often do not distinguish \( Y \) from the image in \( X \) of the given embedding.
Let $\psi : [0, 1] \times Y \to X$ be an isotopy of embeddings $\psi_t : Y \to X$. By Exercise 3.6.5, we can find a function $h : [0, 1] \times Y \to \mathbb{R}$ such that $\psi^*\omega = dt \wedge dh$, i.e.,

$$\psi^*\omega + dt \wedge (-dh) = 0.$$ 

By Theorem 3.6.7, we can find a Hamiltonian $H : [0, 1] \times X \to \mathbb{R}$ such that

$$\psi(t, x) = \phi_H^t(\psi(0, x)) = \phi_H^t(\psi_0(x)).$$ \hfill (3.7.36)

(When the exact Lagrangian isotopy $(h, \psi)$ is given, we can always adjust $H$ away from the support of the isotopy so that $H$ satisfies the normalization condition

$$\int_X H(t)\omega^n = 0.$$

Also, if the isotopy is boundary flat, then $h$ and $H$ can be made boundary flat. Namely, we may assume that there exists $\epsilon_0 > 0$ such that

$$H \equiv 0 \text{ for } t \in [0, \epsilon_0] \cup [1 - \epsilon_0, 1].$$ \hfill (3.7.37)

An exact Lagrangian isotopy $\psi : [0, 1] \times Y \to X$ is called an exact Lagrangian loop if the associated pair $(h, \psi)$ satisfies

$$\psi(0, x) = \psi(1, x), \quad h(0, x) = h(1, x)$$

for all $x \in Y$.

**Definition 3.7.6** Let $\psi : [0, 1] \times Y \to X$ be an exact Lagrangian loop. The *Lagrangian suspension* associated with $(h, \psi)$ is the Lagrangian embedding

$$\iota_{(h, \psi)} : S^1 \times Y \to \mathbb{R} \times S^1 \times X = T^*S^1 \times X$$

defined by

$$\iota_{(h, \psi)}(\theta, y) = (\theta, -h(\theta, x), \psi(\theta, x)).$$ \hfill (3.7.38)

Now we state a proposition that is an essential ingredient in the construction of Lagrangian suspension and its applications.

Consider an isotopy of Hamiltonian paths, i.e., a two-parameter family of Hamiltonian diffeomorphisms

$$(s, \theta) \in [0, 1] \times S^1 \mapsto \psi_s^\theta \in \text{Ham}(X, \omega).$$

We denote by $H = H(s, \theta, x)$ the normalized Hamiltonian generating the $\theta$-isotopy $\theta \mapsto \psi_s^\theta$ and by $K = K(s, \theta, x)$ the one generating the $s$-isotopy $s \mapsto \psi_s^\theta$. In other words,
3.7 Construction of Lagrangian submanifolds

\[ \frac{\partial \psi^\theta}{\partial \theta} \circ (\psi^\theta)^{-1} = X_H, \quad \frac{\partial \psi^s}{\partial s} \circ (\psi^s)^{-1} = X_K. \]

Recall from (2.5.34) the identity

\[ \frac{\partial H}{\partial s} = \frac{\partial K}{\partial \theta} - \{K, H\} = \frac{\partial K}{\partial \theta} + \{H, K\} \quad (3.7.39) \]

for the normalized Hamiltonian \( H \).

**Proposition 3.7.7** Let \((h, \psi)\) be an exact Lagrangian loop. Suppose that the image of \( \psi \) is contained in \( U \subset X \). Equip \( T^*S^1 \times X \) with the symplectic form

\[ d\theta \wedge db + \omega_X, \]

where \((\theta, b)\) is the canonical coordinates of \( T^*S^1 \). Then the Lagrangian embedding \( \iota_{(h, \psi)} : S^1 \times Y \rightarrow T^*S^1 \times X \) is isotopic to the embedding \( o_{S^1} \times i_Y : S^1 \times Y \subset T^*S^1 \times X \) by a Hamiltonian flow supported in \( T^*S^1 \times U \).

We adapt the proof given in the appendix of (Oh07) to prove the following theorem.

**Theorem 3.7.8** Let \( Y \subset X \) be a Lagrangian embedding and consider a two-parameter family \( \psi^t_s \) of Hamiltonian diffeomorphisms as above. Then the isotopy \( f : [0, 1] \times S^1 \times Y \rightarrow T^*S^1 \times X =: M \) defined by

\[ f(s, \theta, x) = (\theta, -H(s, \theta, \psi^s_\theta(x)), \psi^s_\theta(x)) \quad (3.7.40) \]

is an exact Lagrangian isotopy in the sense of Definition 3.6.3. Furthermore, we have

\[ i_s^* \alpha = d(K \circ f)_s. \quad (3.7.41) \]

We would like to mention that here \( s \) replaces the role of \( t \) and \([a, b] \times Y\) that of \( Y \) in Definition 3.6.3.

**Proof** First we compute

\[ f^* (d\theta \wedge db + \omega_X). \]

Since \( o_{S^1} \times i_Y \subset T^*S^1 \times X \) is a Lagrangian embedding and \( \psi^s_\theta \) is symplectic, it is easy to check that

\[ f^* (d\theta \wedge db + \omega_X) = \alpha \wedge ds \]

for some one-form \( \alpha \) on \([0, 1] \times (S^1 \times Y)\). We have

\[ \alpha = \frac{\partial}{\partial s} \big| f^* (d\theta \wedge db + \omega_X), \]
which we now evaluate. First we have

$$\alpha(\xi) = (d\theta \wedge db + \omega_X) \left( \frac{\partial f}{\partial s}, df(\xi) \right)$$

for a general tangent vector $\xi \in T(S^1 \times Y)$. Straightforward computations give rise to the following formulae:

$$df \left( \frac{\partial}{\partial s} \right) = \frac{\partial}{\partial s} \circ \omega^\theta_X \oplus X_K \circ \psi_s^\theta,$$

$$df(\eta) = -d_X H(\psi_s^\theta(\eta)) \frac{\partial}{\partial b} \oplus d\psi_s^\theta(\eta),$$

where $\eta$ is a vector field whose projection to $[0, 1] \times S^1$ is zero. For the first equality, we used the identity $dH(X_K) = \{H, K\}$, and for the second equality, we used (2.5.34). From these formulae, we derive

$$i^*_s \alpha = i^*_s \left( \frac{\partial}{\partial s} \right) \left( d\theta \wedge db + \omega_X \right) = \left( \frac{\partial K}{\partial \theta} \circ f_s \right) \left( d\theta \circ T f_s + d_X K \circ T f_s \right) = f^*_s \left( \frac{\partial K}{\partial \theta} + d_X K \right) = f^*_s dK = d(K \circ f)_s.$$ 

Therefore the form $i^*_s \alpha$ is exact and hence $f$ defines an exact Lagrangian isotopy in the sense of Definition 3.6.3. Furthermore, we have also proved (3.7.41) at the same time. This finishes the proof. 

Here is one easy way of constructing an exact Lagrangian loop from any given exact Lagrangian isotopy. First we introduce the notion of time-reversal of the isotopy.

**Definition 3.7.9** Let $Y \subset X$ be a Lagrangian submanifold and $(h, \psi)$ an exact Lagrangian isotopy. The time-reversal of $(h, \psi)$ is the pair $(\tilde{h}, \tilde{\psi})$ defined by

$$\tilde{\psi}(t, x) = \psi(1 - t, x), \quad \tilde{h}(t, x) = -h(1 - t, x).$$

Obviously, for a boundary flat $(h, \psi)$, the concatenated isotopy

$$(h, \psi) * (\tilde{h}, \tilde{\psi})$$

defines a smooth closed embedding $S^1(2) \times Y \to \mathbb{R} \times S^1(2) \times X$, where we represent $S^1$ by $\mathbb{R}/2\mathbb{Z}$. We call this concatenated isotopy an odd double of the isotopy $(h, \psi)$ and denote it by $(h^{od}, \psi^{od})$.
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Definition 3.7.10 The odd-double suspension of an exact Lagrangian isotopy \((h, \psi) : [0, 1] \times Y \to \mathbb{R} \times X\) is defined by the embedding

\[
\iota(h, \psi) : S^1(2) \times Y \to T^*S^1(2) \times X,
\]

\[
\iota(h, \psi)(\theta, x) = \begin{cases} 
(\theta, -h(\theta, x), \psi(\theta, x)), & \text{for } \theta \in [0, 1], \\
(\theta, h(2 - \theta, x), \psi(2 - \theta, x)), & \text{for } \theta \in [1, 2].
\end{cases}
\]

We also denote the obvious double \(h \ast \tilde{h}\) of the Hamiltonian \(h\) by \(h_{od}\).

This construction of odd-double suspension plays an important role in the proof of the uniqueness result of topological Hamiltonians in (Vi06) and (Oh07).

3.7.3 Chekanov’s suspension

In this section, we explain a different type of suspension introduced by Chekanov (Che96b), which exclusively applies to Lagrangian submanifolds in \(\mathbb{R}^{2n}\).

Consider the submersion \(i_n : S^1 \times \mathbb{R}^n \to \mathbb{R}^{n+1}\) defined by

\[
i_n(\theta, q_1, \ldots, q_n) = (e^{q_1} \cos(2\pi\theta), e^{q_1} \sin(2\pi\theta), q_2, \ldots, q_n),
\]

which defines a diffeomorphism onto \((\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^{n-1}\). Next we denote by \(i_n^* : T^*((\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^{n-1}) \to T^*(S^1 \times \mathbb{R}^n)\) the coderivative of \(i_n\). Then we consider the symplectic embedding

\[
I_n = (i_n^*)^{-1} : T^*(S^1 \times \mathbb{R}^n) \to T^*\mathbb{R}^{n+1} \cong \mathbb{R}^{2n+2}.
\]

Now let \(L \subset \mathbb{R}^{2n} \cong T^*\mathbb{R}^n\) be a Lagrangian submanifold, and let \(a\) be any real number. Denote by \(N_a\) the graph of the parallel one-form \(a \, d\theta\), i.e.,

\[
\{ (\theta, \alpha) \in T^*S^1 \mid \alpha = a \}
\]

in the canonical coordinates of \(T^*S^1\). We regard \(N_a \times L\) as a Lagrangian submanifold of \(T^*S^1 \times T^*\mathbb{R}^n \cong T^*(S^1 \times \mathbb{R}^n)\). Then Chekanov’s suspension of \(L\) is the Lagrangian submanifold defined by

\[
\Theta_a(L) = I_n(N_a \times L) \subset T^*(\mathbb{R}^{n+1}) \cong \mathbb{R}^{2n+2}.
\]

Chekanov (Che96b) used this suspension to obtain the following interesting result.
Theorem 3.7.11 (Chekanov)  Let $S^1 \subset \mathbb{R}^2$ be the standard circle of radius 1 and consider the suspension $\Theta_0(S^1) \subset \mathbb{R}^4$. Then $\Theta_0(S^1)$ is a Lagrangian torus in $\mathbb{R}^4$ that is not Hamiltonian isotopic to the standard torus $S^1 \times S^1$.

This result was originally proved by Chekanov (Che96b) studying a variation of the Ekeland–Hofer capacity. Later Eliashberg and Polterovich (EP97) used a version of one-point open Gromov–Witten invariants to distinguish the two Lagrangian tori.

3.8 The canonical relation and the natural boundary condition

In this section, we study the least action principle on general symplectic manifolds and set up appropriate frameworks for the variational theory of the multi-valued action functional in the context of both open and closed strings.

As in the classical mechanics on the cotangent bundle of Chapter 1, we consider the set of smooth paths

$$\mathcal{P}M = \{\gamma : [0, 1] \to M\}.$$ 

There are natural evaluation maps $\text{ev}_0, \text{ev}_1 : \mathcal{P}M \to M$ given by $\text{ev}_i(\gamma) = \gamma(i)$ for $i = 0, 1$. We combine them to define a map

$$\text{ev} = (\text{ev}_0, \text{ev}_1) : \mathcal{P}M \to M \times M.$$ 

The following concept of canonical relations played an important role in the calculus of Fourier integral operators by Hörmander; see (Hor71) and (GS77).

Definition 3.8.1 (Canonical relation) A subset $C \subset M \times M$ is called a canonical relation if $C$ is a Lagrangian submanifold with respect to the symplectic form $-\omega \oplus \omega$ on $M \times M$.

We will show that the canonical relations play the role of the natural boundary condition in the variational theory.

We define the action one-form denoted by $\alpha$ on $\mathcal{P}M$ by

$$\alpha(\gamma)(\xi) = \int_0^1 \omega(\dot{\gamma}(t), \xi(t))dt.$$ 

The following formula reveals the relation between the canonical relation and the least action principle.
Proposition 3.8.2  Let $\alpha$ be the action one-form on $\mathcal{P}M$. Then we have
\[ d\alpha = e v_1^* \omega - e v_0^* \omega. \]  
(3.8.44)
(Here we regard $\mathcal{P}M$ as an infinite-dimensional manifold.)

Proof  We need to prove
\[ d\alpha(\gamma)(\xi, \eta) = \omega(\xi(1), \eta(1)) - \omega(\xi(0), \eta(0)) \]  
(3.8.45)
for any $\gamma \in \mathcal{P}M$ and $\xi, \eta \in T_\gamma(\mathcal{P}M)$. To compute $d\alpha(\gamma)(\xi, \eta)$, we choose a two-parameter family of paths
\[ V : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to \mathcal{P}M, \]
\[ V(0, 0) = \gamma, \quad \frac{\partial V}{\partial u}|_{(0,0)} = \xi, \quad \frac{\partial V}{\partial s}|_{(0,0)} = \eta. \]
We also denote by $V = V(u, s, t)$ the corresponding map
\[ V : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times [0, 1] \to M. \]
Then we have
\[ d\alpha(\gamma)(\xi, \eta) = \left( \frac{\partial}{\partial u} \alpha \left( \frac{\partial V}{\partial s} \right) - \frac{\partial}{\partial s} \alpha \left( \frac{\partial V}{\partial u} \right) \right)|_{(0,0)} . \]  
(3.8.46)
Here we have used the identity
\[ \left[ \frac{\partial V}{\partial u}, \frac{\partial V}{\partial s} \right] = V_* \left[ \frac{\partial}{\partial s}, \frac{\partial}{\partial u} \right] = 0. \]

Remark 3.8.3  Strictly speaking, we need to be more careful to consider differential forms and their exterior derivatives on the path space $\mathcal{P}M$. Here we suggest that readers take the formula (3.8.46) as the definition of the two-form $d\alpha$ on $\mathcal{P}M$. We refer readers to (GJP91) and (Chen73) for more systematic discussion on the differential forms on the path space.

By definition of the action one-form (3.8.43),
\[ \frac{\partial}{\partial u} \alpha \left( \frac{\partial V}{\partial s} \right) = \int_0^1 \frac{\partial}{\partial u} \left( \omega \left( \frac{\partial V}{\partial t}, \frac{\partial V}{\partial s} \right) \right) dt. \]
Now we fix any torsion-free symplectic connection $\nabla$. Then we have
\[ \frac{\partial}{\partial u} \left( \omega \left( \frac{\partial V}{\partial t}, \frac{\partial V}{\partial s} \right) \right) = \omega \left( \nabla_u \frac{\partial V}{\partial t}, \frac{\partial V}{\partial s} \right) + \omega \left( \frac{\partial V}{\partial t}, \nabla_u \frac{\partial V}{\partial s} \right). \]
By the torsion-free condition on $\nabla$, we have
\[
\nabla_u \frac{\partial V}{\partial s} = \nabla_s \frac{\partial V}{\partial u} \]
and hence the second term will be canceled out by the similar term from 
$(\partial/\partial s)(\partial V/\partial u)$ which will appear in the evaluation of 
$(\partial \alpha/\partial s)(\partial V/\partial u)$ in (3.8.46). Therefore it remains to compute
\[
\omega \left( \nabla_u \frac{\partial V}{\partial t}, \frac{\partial V}{\partial s} \right) = \omega \left( \nabla_s \frac{\partial V}{\partial u}, \frac{\partial V}{\partial s} \right) = \frac{\partial}{\partial t} \left( \omega \left( \frac{\partial V}{\partial u}, \frac{\partial V}{\partial s} \right) \right) - \omega \left( \frac{\partial V}{\partial u}, \nabla_t \frac{\partial V}{\partial s} \right).
\]
Therefore, by interchanging the roles of $u$ and $s$ and using the symplectic
property of $\nabla$, we derive
\[
\frac{\partial}{\partial u} \left( \omega \left( \frac{\partial V}{\partial t}, \frac{\partial V}{\partial s} \right) \right) - \frac{\partial}{\partial s} \left( \omega \left( \frac{\partial V}{\partial t}, \frac{\partial V}{\partial u} \right) \right) = 2 \frac{\partial}{\partial t} \left( \omega \left( \frac{\partial V}{\partial u}, \frac{\partial V}{\partial s} \right) \right) + \omega \left( \frac{\partial V}{\partial s}, \nabla_t \frac{\partial V}{\partial u} \right) - \omega \left( \frac{\partial V}{\partial u}, \nabla_t \frac{\partial V}{\partial s} \right).
\]
Hence we have obtained
\[
d\alpha(\gamma)(\xi, \eta) = \int_0^1 \frac{\partial}{\partial t} \left. \left( \omega \left( \frac{\partial V}{\partial u}, \frac{\partial V}{\partial s} \right) \right) \right|_{(0,0)} dt = \int_0^1 \frac{\partial}{\partial t} \omega(\xi(t), \eta(t)) dt = \omega(\xi(1), \eta(1)) - \omega(\xi(0), \eta(0)),
\]
which is exactly (3.8.45).

For a given subset $C \subset M \times M$, we denote the subset
\[
P_C M = ev^{-1}(C) = \{ \gamma \in \mathcal{P} M \mid (\gamma(0), \gamma(1)) \in C \}.
\]
We will still denote the obvious restriction of $\alpha$ to $P_C M$ by the same
letter $\alpha$. An immediate corollary of the formula (3.8.44) is the following
characterization of the canonical relation as the natural boundary condition.

**Corollary 3.8.4** The action one-form $\alpha$ is closed on $P_C M$ if and only if $C \subset M \times M$ is a canonical relation.
We give here some prominent examples of the canonical relations that will appear in the later chapters on the Floer theory.

**Example 3.8.5**

(i) **(Free loops).** The diagonal $\Delta \subset M \times M$ is a canonical relation. The corresponding path space $\mathcal{P}_\Delta M$ can be naturally identified with the free loop space

$$\mathcal{L}M = \{ \gamma : S^1 \to M \mid S^1 = \mathbb{R}/\mathbb{Z} \}.$$ 

The zero set of $\alpha$ on $\mathcal{P}_\Delta M$ is that of constant paths and so a copy of $M \hookrightarrow \mathcal{P}_\Delta M$ itself.

(ii) **(Twisted loops).** The graph

$$\Delta_{\phi} := \{ (x, \phi(x)) \mid x \in M \}$$

of a symplectic diffeomorphism $\phi : M \to M$ is a canonical relation. The corresponding path space is

$$\mathcal{L}_{\phi} M = \{ \gamma \in \mathcal{P}M \mid \phi(\gamma(0)) = \gamma(1) \}$$

and an element in $\mathcal{L}_{\phi} M$ is called a *twisted loop*. In fact, one can identify a $\phi$-twisted loop with a section of the mapping circle

$$M_{\phi} := [0, 1] \times M / \sim \to S^1$$

where the equivalence relation $\sim$ is provided by

$$(0, x) \sim (1, x') \iff x' = \phi(x).$$

The zero set of $\alpha$ in this case is nothing but the constant loops in $\text{Fix} \phi$.

(iii) **(Lagrangian boundary condition).** Now consider the canonical relation of the form

$$C = L_0 \times L_1,$$

where $L_0$ and $L_1$ are Lagrangian submanifolds of $M$. The zero set of $\alpha$ corresponds to the intersection $L_0 \cap L_1$.

### 3.9 Generating functions and Viterbo invariants

*Generating functions* play an important role in Hamiltonian mechanics and in micro-local analysis, especially in Hörmander’s calculus of Fourier integral operators (Hor71). We refer readers to (Go80) for applications to the classical mechanics and (Hor71), (GS77) for their usage in the theory of Fourier
integral operators. In the geometric point of view, whenever a family of functions on a given manifold $N$ is given, it is associated with a natural (immersed) Lagrangian submanifold on $T^*N$ under a suitable transversality hypothesis. This construction can be put into a particular case of a more general push-forward operation in the calculus of Lagrangian submanifolds. This calculus of Lagrangian submanifolds has recently attracted much attention in relation to functorial construction of morphisms and operations in the Fukaya category (WW10). In this section, we illustrate this calculus applied to a construction of the generating functions.

We start with the following lemma in linear algebra on which the push-forward operation of Lagrangian submanifolds is based. Recall from Lemma 3.2.2 that for any coisotropic subspace $C$ of a symplectic vector space $(V, \Omega)$ the quotient $C/C^\Omega$ carries a canonical symplectic form induced from $\Omega$. (See Lemma 3.2.2.) The next lemma shows under what conditions a Lagrangian subspace $L \subset (V, \Omega)$ canonically induces a Lagrangian subspace on $C/C^\Omega$.

**Lemma 3.9.1** Let $C \subset (V, \Omega)$ be a given coisotropic subspace. Consider a Lagrangian subspace $L \subset (V, \Omega)$ and the intersection $L \cap C$. Suppose

\[ L + C = V. \tag{3.9.47} \]

Then $\pi_C(L \cap C) \subset C/C^\Omega$ is Lagrangian.

**Proof** It follows that $\pi_C(L \cap C)$ is always isotropic in $C/C^\Omega$. Furthermore, (3.9.47) implies $L \cap C^\Omega = L^\Omega \cap C^\Omega = 0$. Therefore, we have $\dim L \cap C = \dim \pi_C(L \cap C)$. It remains to show that $\dim L \cap C = \frac{1}{2} \dim(C/C^\Omega)$. We have $\dim C + \dim C^\Omega = 2n (= \dim V)$ and so

\[ \dim(C/C^\Omega) = 2n - 2 \dim C^\Omega. \]

On the other hand, we have

\[ \dim L \cap C = \dim L + \dim C - 2n = \dim C - n = (2n - \dim C^\Omega) - n \]

\[ = n - \dim C^\Omega = \frac{1}{2} \dim(C/C^\Omega), \]

which finishes the proof. \[ \Box \]

Now we consider a fiber bundle $\pi : E \to M$ and let $S : E \to \mathbb{R}$ be a smooth function on the fiber bundle $E$. We consider $\text{Graph}(dS) \subset T^*E$, which is a Lagrangian submanifold. We would like to push-forward $\text{Graph}(dS)$ to a
Lagrangian submanifold in $T^*N$. Consider a canonical vertical tangent bundle $VTE \subset TE$ and hence the subbundle

$$(VTE)^\perp \subset T^*E,$$

(3.9.48)

where $(VTE)^\perp$ is the subbundle of $T^*E$ whose fiber at $e$ $(VTE)^\perp \subset T^*_e E$ is the annihilator of $VT_e E$. We note that $\dim (VTE)^\perp = \dim T^*_e N$. By choosing a complement of $VT_e E \subset T_e E$, denoted by $HT_e E$, we have an isomorphism

$$\pi^*_h : T^*_e N \rightarrow (HT_e E)^* \rightarrow (VTE)^\perp,$$

where the first map is the dual map of the horizontal projection $\pi_h : HT_e E \rightarrow T^*_e N$.

**Exercise 3.9.2** Define this isomorphism and prove that its definition does not depend on the choice of the complement $HT_e E$ in the splitting $T_e E = VT_e E \oplus HT_e E$.

Therefore we have a natural map

$$\text{Graph}(dS) \cap (VTE)^\perp \rightarrow T^*N.$$

(3.9.49)

We denote by $L_S \subset T^*N$ its image. To give a more concrete description of $L_S$, we recall the notion of the fiber derivative of $S$, denoted by $d^x S : E \rightarrow (VTE)^*$, which is defined by

$$d^x S(e) = dS(e)|_{VT_e E}, \ e \in E.$$

Then we can rewrite $\text{Graph}(dS) \cap (VTE)^\perp = \{(e, D^h S(e)) \mid d^x S(e) = 0\}$, where

$$D^h S(e) := dS(e)|_{HT_e E}.$$

**Proposition 3.9.3** Suppose that the section $d^x S : E \rightarrow (VTE)^*$ is transverse to the zero section $o_{(VTE)^*}$. Then the following statements hold.

(1) The fiber critical point set

$$\Sigma_S := \{e \in E \mid d^x S(e) = 0\}$$

is a smooth manifold.

(2) The image of an immersion $i_S : \Sigma_S \rightarrow T^*N$

$$L_S = \{(q, dS|_{\Sigma_S}(e)) \mid e \in \Sigma_S, \pi(e) = q\}$$

is Lagrangian.

If an (immersed) Lagrangian submanifold $L \subset T^*N$ can be represented by $S$ as above, we call $S$ a generating function of $L$. 
Proof By definition of $\Sigma_S$, the first statement follows from the transversality theorem.

For the second statement, if we denote $e = (q, \xi)$ with $\pi(e) = q$ and $\xi \in VT_e E$, the map $i_S : \Sigma_S \to T^* M$ is given by

$$i_S(e) = (\pi_E(e), D^hS(e) \circ (\pi_h(e))^{-1}), \ e \in \Sigma_S.$$ 

Now we compute

$$i^*_S(\theta)(e)(v) = \theta(di_S(e)(v)) = D^hS(e)(v) = i^*_S \omega_0,$$

where $\theta = pdq$ is the Liouville one-form on $T^* M$. This proves that $i^*_S(\theta) = i^*_S \omega_0 = 0$. Therefore it remains to show that $i_S$ is an immersion.

For this, suppose that $di_S(e)(v) = 0$ for $e \in \Sigma_S$ and $v \in T_e \Sigma_S$. Then we have $d\pi_E(e)(v) = 0$ and so $v$ is tangent to the fiber of $\pi_E : E \to N$. On the other hand, $v$ is also tangent to $\Sigma_S$. Therefore, by differentiating $d^S(e) = 0$ along $v \in T_e \Sigma_S$, we obtain $d(d^S(e))(v) = 0$. Since $v$ is vertical, $di_S(v)(v) = 0$ implies that $d^v(D^hS)(e)(v) = 0$. Therefore we obtain $d^v(d^S(e))(v) = 0$. But the transversality hypothesis of $d^S : E \to (VTE)^*$ to the zero section of $(T^vE)^*$ is equivalent to the statement that $d^v(d^S)(e) : VT_e E \to (VT_e E)^*$ is an isomorphism at any point $e \in \Sigma_S$. Therefore we have proved $v = 0$. This proves that $i_S$ is an immersion and finishes the proof. \[\Box\]

Remark 3.9.4 When $N = \mathbb{R}^n$ and $E = \mathbb{R}^n \times \mathbb{R}^m$, and $S = S(q, \xi)$ is a function on $\mathbb{R}^n \times \mathbb{R}^m$, we can identify

$$d^S = d_\xi S, \quad D^hS = d_q S$$

and $\Sigma_S = \{(q, \xi) \mid d_\xi S(q, \xi) = 0\}$. Furthermore, we have $VTE \cong \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ and $d^S : E \to (VTE)^*$ has the expression

$$d^S(q, \xi) = (q, \xi, d^S(q, \xi)).$$

Therefore transversality of $d^S$ against the zero section

$$VTE \cong \mathbb{R}^n \times \mathbb{R}^m \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^m \times (\mathbb{R}^m)^*$$

at $e = (q, \xi) \in \Sigma_S$ is equivalent to the invertibility of the Hessian $\text{Hess}(S|_{E_q})$.

An immediate corollary of the proof of Proposition 3.9.3 is as follows.
Corollary 3.9.5  Any Lagrangian immersion in $T^*N$ generated by a generating function is exact.

It turns out that not all exact Lagrangian immersion is generated by a generating function. See (Lat91) for some detailed study of this obstruction to representability of exact Lagrangian submanifolds by generating functions. In relation to the existence of generating functions of an exact Lagrangian submanifold, the following theorem of Laudenbach and Sikorav, Theorem 2.1 (LS85), is an important existence result.

Theorem 3.9.6 (Laudenbach–Sikorav)  Let $L \subset T^*N$ be a Hamiltonian deformation $L = \phi^1_H(o_N)$ of the zero section $o_N \subset T^*N$. Then there exists a finite-dimensional vector bundle $E \to N$ and a smooth function $S : E \to \mathbb{R}$ representing $\phi^1_H(o_N)$, which is quadratic at infinity in addition.

Here $S$ is said to be quadratic at infinity if there exists a nondegenerate fiberwise quadratic form $Q$ such that $S(q, \xi) = Q(q, \xi)$ outside a compact subset of $E$. Motivated by this theorem, we restrict the discussion to this special class of generating functions. Following Viterbo (Vi92), we call a generating function quadratic at infinity by GFQI.

For a given fiberwise quadratic form $Q$ on $E$, we denote by $S(Q,E)$ the set of GFQI with $S(q, \xi) = Q(q, \xi)$ outside a compact subset of $E$. We note that when $E \to N$ is a vector bundle we have the canonical isomorphism $VT_eE \cong E_{\pi(e)}$ and hence can identify $VTE \equiv \pi_E^*E$ and $(VTE)^* \equiv \pi_E^*E^*$, where $\pi_E : E \to N$ is the projection. We define

$$S^0_{(Q,E)} := \{ S \in S_{(Q,E)} \mid d^*S \text{ is transverse to } o_{\pi_E E^*} \}.$$  

It is easy to see that $S^0_{(Q,E)}$ is a dense subset of $S_{(Q,E)}$ in the compact open topology.

We now introduce two operations on generating functions that do not change their generated Lagrangian submanifolds.

Definition 3.9.7

(1) [Stabilization] We call $\tilde{S} : E \oplus F \to \mathbb{R}$ a stabilization of $S$ if

$$\tilde{S}(q, \xi, \eta) = S(q, \xi) + Q(q, \eta) + C$$

for some nondegenerate fiberwise quadratic form $Q$ on a vector bundle $F \to N$ and some constant $C$. 

(2) [Gauge equivalence] We say $S_1, S_2 : E \to \mathbb{R}$ are gauge equivalent if

$$S_2 = S_1 \circ \Phi$$

for some fiber-preserving diffeomorphism $\Phi : E \to E$.

The following uniqueness result is a fundamental theorem proved in (Vi92) and (Th99), to whose proof we refer readers.

**Theorem 3.9.8** (Viterbo) *The generating function in Theorem 3.9.6 is unique up to the stabilization and the gauge equivalence.*

Now let a GFQI $S : E \to \mathbb{R}$ be given. Denote the sub-level sets of $S \in \mathcal{S}_{(Q;E)}$ by

$$E^\lambda := \{ e \in E \mid S(e) \leq \lambda \}.$$

For $|\lambda|$ large enough, the homotopy type of

$$E^\lambda = \{ e \in E \mid S(e) \leq \lambda \} = \{ e \in E \mid Q(e) \leq \lambda \}$$

is independent of $\lambda$. For such $\lambda$, we denote $E^\lambda$ by $E^\infty$ and $E^{-\lambda}$ by $E^{-\infty}$. Note that

$$(E^\infty, E^{-\infty}) \cong (D(E^-), S(E^-)),$$

where $E^-$ denotes the negative bundle of the quadratic form $Q$, and $D(E^-)$ and $S(E^-)$ denote the disc and the sphere bundle associated with $E^-$, respectively.

The Thom isomorphism provides the isomorphism

$$H^*(M) \cong H^*(D(E^-), S(E^-)) \cong H^*(E^\infty, E^{-\infty}). \quad (3.9.50)$$

In (Vi92), Viterbo associated with each $(u, S) \in H^*(M) \times \mathcal{S}^0_{(Q;E)}$ the real number

$$c(u, S) = \inf \{ \lambda \mid j^*_\lambda Tu \neq 0 \}.$$

Here $T$ denotes the isomorphism (3.9.50) and

$$j^*_\lambda : H^*(E^\infty, E^{-\infty}) \to H^*(E^\lambda, E^{-\infty})$$

is the homomorphism induced by the inclusion

$$j_\lambda : E^\lambda \to E^\infty.$$

Following (MiO95), we generalize this construction by considering arbitrary closed submanifold $K \subset N$, denote by $\pi|_K : E|_K \to K$ the restriction of a bundle $E$ to $K$. 
Definition 3.9.9 Let $S \in S^0_{(Q,E)}$ and $K \subset N$ be given. For each $0 \neq u \in H^*(K)$, we define

$$c(S; u, K) := \inf\{\lambda | j^*_u T u \neq 0 \text{ in } H^*(E^1_K, E^{-\infty})\}.$$ 

The following result can be proved either in the way how Viterbo did in (Vi92) or in the Floer-theoretic way as was done in (Oh97b).

Theorem 3.9.10 (Compare with Theorem 2.3 of (Vi92)) Let $S \in S^0_{(Q,E)}$ be a generating function for a Lagrangian submanifold $L \subset T^* N$, and let $K \subset N$ be a closed submanifold. Then the following apply.

1. $c(S; u, K)$ is a critical value of $S|_{E^0}$.
2. For each $0 \neq u \in H^*(K)$ there exists a point $x_u \in \nu^* K \cap L$ such that

$$c(S; u, K) - c(S; v, K) = \int_{\gamma} \theta,$$

where $\gamma$ is any smooth path in $L$ connecting $x_u$ to $x_v$.
3. Let $1$ and $\mu_N$ be the generators of $H^0(N)$ and $H^n(N)$, respectively. Denote

$$\gamma(L) := c(\mu_N, S) - c(1, S).$$

Then $\gamma(L) \geq 0$ and $\gamma(L) = 0$ if and only if $L = o_M$.
4. $S \mapsto c(S; u, K)$ is a continuous function in compact open topology of $S$. 
Symplectic fibrations

The precise notion of symplectic fibrations was introduced by Guillemin, Lerman and Sternberg (GLS96) in relation to the geometry of moment maps and the representation theory. In particular, they spelled out the precise notion of Hamiltonian fibrations as the subclass of symplectic fibrations which enables one to define a closed extension of the given fiberwise symplectic structures. They called the latter closed extension the coupling form. A simple form of this notion of Hamiltonian fibrations was implicit when Floer formulated the celebrated Floer homology of symplectic fixed points in terms of the mapping cylinder (Fl87). Seidel (Se97) related the Hamiltonian fibrations to the study of Floer homology and its product structure in a systematic way. Ever since then interactions between the Hamiltonian fibrations and Floer homology have been a crucial element in the recent development of the chain-level Floer theory and its applications to symplectic topology, (En00), (Schw00), (Oh05b), (Oh05c), (Oh05d), (Oh06a), (Oh07) and (Oh09a), and to mirror symmetry, (Se03a) and (Se03b). In this chapter, we will review the basic elements of the theory of Hamiltonian fibrations and their symplectic connections, emphasizing their relations to Hamiltonian dynamics and perturbed Cauchy–Riemann equations.

4.1 Symplectic fibrations and symplectic connections

Roughly speaking, a symplectic fiber bundle $\pi : E \to B$ with fiber modeled with a manifold $F$ is a fiber bundle whose structure group is the group $\text{Symp}(F, \omega_F)$ for a symplectic form $\omega_F$ on $F$. Here is the obvious elementary definition.
**Definition 4.1.1** A fiber bundle \( \pi : E \to B \) with fiber modeled by a symplectic manifold \((F, \omega_F)\) is called a *symplectic fiber bundle* if there exists an open covering \( \{ U_\alpha \} \) of \( B \) and diffeomorphisms

\[
\Phi_\alpha : \quad \pi^{-1}(U_\alpha) \to U_\alpha \times F
\]

such that the transition maps \( \Psi_{\alpha\beta} \)

\[
\Psi_{\alpha\beta} = \Phi_\alpha \circ \Phi_\beta^{-1}
\]

have the form

\[
\Psi_{\alpha\beta}(b, p) = (b, \varphi_{\alpha\beta}(b, p)),
\]

where \( \varphi_{\alpha\beta}(b, \cdot) : F_b = \pi^{-1}(b) \to F \) is a symplectic diffeomorphism for all \( b \in U_\alpha \cap U_\beta \).

Now let us analyze some consequences that follow from this definition. Let \( b \in U_\alpha \subset B \) and \( \Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F \). We denote by \( \pi_2 : U_\alpha \times F \to F \) the projection. Then the fiber \( \pi^{-1}(b) = F_b \) carries a symplectic form

\[
\omega_b = \Phi_\alpha^*(\pi_2^*\omega_F).
\]

**Proposition 4.1.2** Let \( b \in B \). The form \( \omega_b \) defined above is independent of the choice of \( \Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F \). In other words, we have

\[
\Phi_\alpha^*(\pi_2^*\omega_F) = \Phi_\beta^*(\pi_2^*\omega_F).
\]

**Proof** The proposition follows immediately from the condition that \( \varphi_{\alpha\beta}(b, \cdot) \in \text{Symp}(F, \omega_F) \) in (4.1.2) is a symplectic diffeomorphism.

In terms of this proposition, each symplectic fiber bundle over \( B \) can be regarded as a \( B \)-family of symplectic manifolds \((F_b, \omega_b), b \in B\), that is locally trivial as a symplectic manifold.

**Exercise 4.1.3** Prove the above statement: any locally trivial fibration of symplectic manifolds over \( B \) is a symplectic fiber bundle.

Let \( \pi : E \to B \) be a symplectic fibration with fiber \((F, \omega_F)\). We denote by \( F_b = \pi^{-1}(b) \) and \( \omega_b \) the symplectic form defined above. At each point \( e \in E \), we denote by

\[
VT_e E = \ker d\pi(e) \cong T_e F_{\pi(e)}
\]
the vertical tangent space of the fiber, and $VTE = \bigcup_{e \in E} VT_e E$. Recall that an (Ehresman) connection $\Gamma$ on $\pi : E \to B$ is a choice of horizontal subspaces $HT_e E$

$$\Gamma : TE = VTE \oplus HTE, \quad HTE = \bigcup_{e \in E} HT_e E.$$ 

Given such a connection, each path $\gamma : [0, 1] \to B$ induces a horizontal vector field on $\gamma^* E$ denoted by $\xi_\gamma$. More precisely, the value $\xi_\gamma(e)$ at each $e \in (\gamma^* E)_t = E_{\gamma(t)}$ is the unique horizontal lift of tangent vector $\gamma'(t)$ that satisfies

$$d\pi(\xi_\gamma(e)) = \gamma'(t).$$

By solving the ordinary differential equation

$$\dot{e}(t) = \xi_\gamma(e(t)), \quad e(0) = e \quad (4.1.3)$$

for each given $e \in F_{\gamma(0)}$, and evaluating the solution $e(t)$ at $t = 1$, we obtain a diffeomorphism $\phi_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$, called the holonomy of the connection $\Gamma$ along $\gamma$.

**Remark 4.1.4** Such a diffeomorphism always exists if the fiber $F$ is compact. However, if $F$ is non-compact, one needs to require some tame behavior of the connection $\Gamma$ at infinity of the fibers $F_b$ enabling one to solve the ODE (4.1.3) uniformly over the interval $0 \leq t \leq 1$ for all initial data $e \in F_{\gamma(0)}$.

**Definition 4.1.5** A connection $\Gamma$ of $\pi : E \to B$ is called a symplectic connection if all holonomy $\Pi_\gamma$ defines a symplectic diffeomorphism, i.e.,

$$\Pi_\gamma^* \omega_{\gamma(1)} = \omega_{\gamma(0)}.$$ 

Now we would like to describe a geometric condition for $\Gamma$ to be symplectic. We consider a 2-form $\Omega$ on $E$ that is compatible with the symplectic fibration $\pi : E \to B$, i.e.,

$$\iota_b^* \Omega = \omega_b$$

at each $b \in B$, where $\iota_b : F_b \hookrightarrow E$. Nondegeneracy of $\omega_b$ implies that the field of subspaces

$$(VT_e E)_\Omega = \{ \xi \in T_e E \mid \Omega(\xi, \eta) = 0 \forall \eta \in VT_e E \}$$

defines a connection $\Gamma_\Omega$. Conversely, every horizontal distribution comes this way but the choice of 2-form $\Omega$ is not unique. Two compatible 2-forms $\Omega_1$ and $\Omega_2$ define the same horizontal distributions if and only if

$$VTE \subset \ker(\Omega_1 - \Omega_2).$$
The following result from (GLSW83) characterizes a symplectic connection $\Gamma$ in terms of the defining 2-form $\Omega$.

**Proposition 4.1.6** Suppose that a 2-form $\Omega$ on $E$ is compatible with the symplectic fibration $\pi : E \to B$, i.e., $i_b^*\Omega = \omega_b$ at all $b \in B$. Then the connection $\Gamma_\Omega$ is symplectic if and only if the form $\Omega$ is vertically closed in the sense that

$$d\Omega(e, \xi_1, \xi_2, \cdot) = 0$$

for all $e \in E$ and $\xi_1, \xi_2 \in V_T e$.

**Proof** Suppose that $\Gamma$ is symplectic. For an arbitrary vector field $\xi$ and vertical tangent vector fields $\xi_1, \xi_2$ on $E$ we evaluate $d\Omega(e, \xi_1, \xi_2, \cdot)$ at a point $e \in E$,

$$d\Omega(e, \xi_1, \xi_2, \xi) = \xi_1[\Omega(\xi_2, \xi)] + \xi_2[\Omega(\xi_1, \xi)] + \xi[\Omega(\xi_1, \xi_2)]$$

$$-\Omega([\xi_1, \xi_2], \xi) - \Omega(\xi_2, \xi_1) - \Omega(\xi_1, \xi_2).$$

We first consider the case in which $\xi$ is horizontal, i.e., $\xi = \eta^\sharp$ is the horizontal lift of the vector field $\eta$ on $B$. Then, by the compatibility of $\Omega$ to $\Gamma$, the above reduces to

$$d\Omega(\xi_1, \xi_2, \xi) = \xi[\Omega(\xi_1, \xi_2)] - \Omega([\xi_2, \xi], \xi_1) - \Omega([\xi, \xi_1], \xi_2).$$  (4.1.4)

On the other hand, we have for $(\mathcal{L}_\eta^\sharp \Omega)(e)$ the formula

$$(\mathcal{L}_\eta^\sharp \Omega)(e)(\xi_1, \xi_2) = \eta^\sharp[\Omega(\xi_1, \xi_2)] - \Omega([\xi_2, \eta^\sharp], \xi_1) - \Omega([\eta^\sharp, \xi_1], \xi_2)$$  (4.1.5)

for $\xi_1, \xi_2 \in T_e E$. If we denote by $\phi^t_\eta$ and $\phi^t_\eta^\sharp$ the flows of $\eta^\sharp$ on $E$ and of $\eta$ on $B$, respectively, then

$$\mathcal{L}_\eta^\sharp \Omega(e) \bigg|_{VTE} = \left. \frac{d}{dt} \right|_{t=0} (\phi^t_\eta)^* \Omega(e) \bigg|_{VTE} = \left. \frac{d}{dt} \right|_{t=0} \left( \Pi_t^\gamma \right)^{-1}(\omega_\gamma(0)) (e),$$

where $b = \pi(e)$ and $\gamma$ is the curve $t \mapsto \phi_t^\sharp(b)$. But the latter is precisely $\nabla_\eta \omega_b(e)$ by the definition of covariant derivatives and hence

$$\mathcal{L}_\eta^\sharp \Omega(e) \bigg|_{VTE} = \nabla_\eta \omega_b(e).$$  (4.1.6)

Since $\Gamma$ is symplectic, $\nabla \omega_b = 0$ and so $(\mathcal{L}_\eta^\sharp \Omega)(\xi_1, \xi_2) = 0$. Combining these, we have finished the proof of $d\Omega(\xi_1, \xi_2, \xi) = 0$ for a horizontal vector field $\xi$.

Now assume that $\xi$ is vertical. Then we have

$$d\Omega(e)(\xi_1, \xi_2, \xi) = d\omega_b(\xi_1(e), \xi_2(e), \xi(e))$$

for $b = \pi(e)$. Since $\omega_b$ is closed on $E_b$ and $\xi_1, \xi_2$ are vertical, the right-hand side vanishes. This proves $d\Omega(\xi_1, \xi_2, \cdot) = 0$ for all vertical $\xi_1, \xi_2$. 


Next we prove the converse. We need to show that the parallel-transport map along any curve $\gamma : [0, 1] \to B$ is symplectic. In other words, we shall show that
\[ \Pi'_\gamma(\omega_{\gamma(1)}) = \omega_{\gamma(0)}. \]
We will in fact prove the stronger statement
\[ (\Pi_{\gamma,t})^*(\omega_{\gamma(t)}) = \omega_{\gamma(0)} \]
for all $t$, which is equivalent to
\[ 0 = \frac{d}{dt}(\Pi_{\gamma,t})^*(\omega_{\gamma(t)}) = \nabla_{\gamma(t)}'\omega_{\gamma(t)}. \]
But (4.1.6) implies that
\[ \nabla_{\gamma(t)}\omega_{\gamma(t)}(\xi_1(\gamma^t(t)), \xi_2(\gamma^t(t))) = \mathcal{L}_{\eta^t}\Omega(\gamma^t(t))(\xi_1(\gamma^t(t)), \xi_2(\gamma^t(t))) \]
for any horizontal vector field $\eta^t$ with $\eta^t(\gamma^t(t)) = (\gamma^t)'(t)$. Using (4.1.4) and (4.1.5), we then compute
\[ \mathcal{L}_{\eta^t}\Omega(\gamma^t(t))(\xi_1(\gamma^t(t)), \xi_2(\gamma^t(t))) = d\Omega(\gamma^t(t))(\xi_1, \xi_2, (\gamma^t)'(t)), \]
which vanishes by the hypothesis that $\Omega$ is vertically closed. $\square$

One can restate the above vertical closedness of $\Omega$ as the closedness of the form $\gamma^*\Omega$ on $\gamma^*E$ for any smooth loop $\gamma : S^1 \to B$. In the next section, we will study the symplectic fibration $\pi : E \to B$ with symplectic connection generated by a closed compatible 2-form $\Omega$ which plays an important role in the topological quantum-field-theory formulation of Floer homology in the literature (Schw95, Se03a, L04).

### 4.2 Hamiltonian fibration

In relation to the Floer theory, a special type of symplectic fibration plays an important role. In this section, we discuss this particular fibration, called Hamiltonian fibration, partly following the exposition in (GLS96) and (MSa94) to which we refer readers for a more detailed exposition. The choice of materials made for this section is strictly governed by the needs later in the construction of the pants product in Floer cohomology and in the proof of the triangle inequality of spectral invariants under the product.

**Definition 4.2.1** A symplectic fibration $\pi : E \to B$ is called Hamiltonian if it carries a 2-form $\Omega$ compatible to $\pi$ that is closed.
The following theorem underpins the essence of Hamiltonian fibrations in the midst of symplectic fibrations. It is quite a remarkable theorem in that at first sight one finds that the notion of ‘Hamiltonian’ is rather un-geometric. On the other hand, this theorem provides a very geometric characterization of what the Hamiltonian property represents in this context of symplectic fibrations. The proof given here is rather different from those given in (GLS96) and in (MSa94). In this proof we avoid using too much background material but go directly to the proof using only the first principles of differential calculus and exploiting the characterization of symplectic fibrations over the circle and over the surface provided in Examples 4.2.3 and 4.2.5, respectively.

**Theorem 4.2.2** Let $\pi : E \to B$ be a symplectic fibration, and let $\Gamma$ be a symplectic connection on $E$. Then the following statements are equivalent.

1. There exists a closed compatible 2-form $\Omega$ such that $\Gamma = \Gamma_\Omega$.
2. The holonomy of $\Gamma$ around any contractible loop in $B$ is Hamiltonian.

Moreover, if (2) holds, then there exists a unique closed compatible 2-form $\Omega_\Gamma$ on $E$ that generates $\Gamma$ and satisfies the normalization condition

$$\pi^! (\Omega_\Gamma^{k+1}) = 0 \text{ in } \Omega^2(B),$$

where $2k$ is the dimension of the fiber and $\pi^! (\Omega_\Gamma^{k+1})$ is the integration of $\Omega_\Gamma^{k+1}$ along the fiber. Any other closed compatible 2-form $\Omega$ on $E$ that generates $\Gamma$ is related to $\Omega_\Gamma$ by

$$\Omega - \Omega_\Gamma = \pi^* \beta$$

for some closed 2-form $\beta$ on $B$ that is uniquely determined by $\Omega$.

The 2-form $\Omega_\Gamma \in \Omega^2(E)$ above is called the *coupling form* of $\Gamma$.

Before giving the proof of this theorem, we illustrate the notion by some examples naturally arising from Hamiltonian dynamics.

**Example 4.2.3** (Mapping circle) Consider a symplectic manifold $(M, \omega)$ and a symplectic diffeomorphism $\phi : M \to M$. We form the *mapping circle*, denoted by $M_\phi$,

$$E = M_\phi := \{(t, x) \in [0, 1] \times M\}/\sim$$

where $\sim$ is the equivalence relation defined by

$$(1, \phi(x)) \sim (0, x).$$
A more dynamical description of this equivalence relation is the following. Consider the action on $\mathbb{R} \times M$ induced by the discrete group generated by

$$(t, x) \mapsto (1 + t, \phi(x)).$$

This action is free and its quotient is precisely $M_\phi$. We have the natural projection

$$M_\phi \to \mathbb{R}/\mathbb{Z} \cong S^1; \ [(t, x)] \to t.$$

To see the local triviality of this fibration, we consider the open covering

$$\mathbb{R}/\mathbb{Z} = U_\alpha \cup U_\beta; \ U_\alpha = [(-1/4, 3/4)], \ U_\beta = [(1/4, 5/4)].$$

and the trivializations

$$\Phi_\alpha : M_\phi|_{(-1/4, 3/4)} \to (-1/4, 3/4) \times M; \ [(t, x)] \to (t, x)$$

and

$$\Phi_\beta : M_\phi|_{(1/4, 5/4)} \to (1/4, 5/4); \ [(t, x)] \to (t, \phi(x)).$$

Note that $U_\alpha \cap U_\beta = (1/2, 1)$. The transition map $\Phi_{\alpha\beta} : (1/2, 1) \times M \to (1/2, 1) \times M$ is given by $\Phi_{\alpha\beta}(t, x) = (t, \phi(x))$, which is symplectic since $\phi$ is symplectic.

Next we define a symplectic connection of $\pi : E = M_\phi \to S^1$. First we consider the linearization of the action $(t, x) \mapsto (t + 1, \phi(x))$ on $T([0, 1] \times M)$, which is given by

$$((t, x), (a, \xi)) \mapsto ((t + 1, \phi(x)), (a, d\phi(x)(\xi)))$$

for $((t, x), (a, \xi)) \in T([0, 1] \times M)$. Then the tangent bundle $TM_\phi$ is realized as the quotient of $T([0, 1] \times M)$ by this action. We have the vertical part given by

$$T^vM_\phi = \{((t, x), (a, \xi)) \mid a = 0\}/\sim$$

and so a lift of the vector $\partial/\partial t$ has the form

$$\frac{\partial}{\partial t} \oplus X_t(x)$$

where $x \mapsto X_t$ is a vector field on $E_t$. For this lifting to be consistent with the equivalence relation $\sim$, it should satisfy

$$X_{t+1}(\phi(x)) = d\phi_x(X_t(x)), \quad \text{or equivalently } \phi^*X_{t+1} = X_t.$$

The holonomy $\phi : E_{t_0} \to E_{t_1}$ along the interval $[t_0, t_1] \subset S^1$ is obtained by solving the ODE

$$\frac{de}{dt} = \overline{X}(e(t))$$
4.2 Hamiltonian fibration

for a curve $e : [t_0, t_1] \to E$, where $\bar{X}$ is the horizontal lift of $\partial/\partial t$ over $[t_0, t_1]$. In the above trivialization, if we write $e(t) = (t, \gamma(t))$ and $\bar{X}(e(t)) = (\partial/\partial t, X_t(\gamma(t)))$, $\gamma$ satisfies $\dot{\gamma}(t) = X_t(\gamma(t))$. Then the holonomy is symplectic if and only if the vector field $X_t$ satisfies $L_{X_t} \omega = 0$, i.e., $d(X_t \lfloor \omega) = 0$, i.e., $X_t$ is a (time-dependent) symplectic vector field. In conclusion, a symplectic connection on $M_\phi \to S^1$ is equivalent to a choice of a one-parameter family of the symplectic vector field $\{X_t\}_{t \in [0, 1]}$ that satisfies

$$\phi^* X_{t+1} = X_t.$$  

We note that such a vector field always exists since we can always put $X_t = 0$ near $t = 0, 1$ and extending periodically over $\mathbb{R}$. Next it is easy to see that the family $\{X_t\}$ gives rise to a Hamiltonian connection if and only if $X_t$ is Hamiltonian, i.e., $X_t = X_{H_t}$ for some time-dependent function $H_t$. Using the fact that $\phi$ is symplectic, it follows that $H_t$ satisfies

$$\phi^*(dH_{t+1}) = dH_t.$$  

Next we consider the closed 2-form $\Omega = \omega + d(H_t dt)$ on $\mathbb{R} \times M$. By definition, it restricts to $\omega$ on each fiber. If we apply the push-forward of $\Omega$ under the action $(t, x) \to (t + 1, \phi(x))$, we obtain $\phi^* \omega + \phi^* d(H_{t+1} dt) = \omega + d(H_t dt)$, i.e., the form $\Omega$ descends to a closed 2-form $M_\phi$. Since $S^1$ is one-dimensional, the normalization required for $\Omega$ is vacuous. Therefore

$$\Omega = \omega + d(H_t dt)$$  

is the unique coupling form associated with the Hamiltonian connection.

Conversely, we have the following.

**Theorem 4.2.4** Consider any symplectic fibration $E \to S^1$ with its fiber $(M, \omega)$. Then there exists a symplectic diffeomorphism $\phi$ and an isomorphism $\Phi : E \to M_\phi$ as a symplectic fibration.

**Proof** We pick a symplectic connection $\Gamma$ on $E$ and consider the parallel-transport map $\phi_{0,t} : E_0 \to E_t$, where $E_t = \pi^{-1}(t)$. This defines a trivialization $\Psi_+ : (-1/4, 3/4) \times E_0 \to E|_{(-1/4,3/4)}$ by $\Psi_+(t, e_0) = \phi_{0,t}(e_0)$ and $\Psi_- : (1/4, 5/4) \times E_1$ by $\Psi_-(t, e_1) = \phi_{1,t}(e_1)$, where we note that $E_1 = E_0$. By definition, we have

$$\Psi_-^{-1} \circ \Psi_+(t, e_0) = \Psi_-^{-1}(\phi_{0,t}(e_0)).$$
Noting that \(\phi_{0,t}(e_0) = \phi_{1,t}(\phi_{t,1}\phi_{0,t}(e_0)) = \phi_{1,t}(\phi_{0,1}(e_0))\), we obtain 
\[\Psi_+^{-1} \circ \Psi_+(t, e_0) = (t, \phi_{0,1}(e_0))\]
on \((1/4, 3/4) \times E_0\). Therefore, if we set \(\Phi_+ = \Psi_+^{-1}\) on \(E|_{(1/4, 3/4)}\) and \(\Phi_- = \Psi_-^{-1}\) on \(E|_{(1/4, 5/4)}\), we obtain a trivialization that is exactly the same as the one in the description of a mapping circle with \(\phi = \phi_{0,1}\) the holonomy of the connection \(\Gamma\) at \(0 \in \mathbb{R}/\mathbb{Z}\). This finishes the proof. \(\Box\)

The following example will be important in relation to the Floer homology.

**Example 4.2.5** (Mapping cylinder) We consider an isotopy of symplectic diffeomorphisms \(\{\phi^s\}_{s \in [0, 1]}\) and perform the above mapping circle construction at each time \(s \in [0, 1]\). We obtain a symplectic fibration \(E \to [0, 1] \times S^1\). A symplectic connection on \(E\) is given by a two-parameter family of symplectic vector fields \(\{X_{(s,t)}\}\) satisfying \((\phi^s)^*X_{(s,t+1)} = X_{(s,t)}\) with \(X_{s,t} = 0\) for \(s = 0, 1\).

When the connection is Hamiltonian it will be given by a two-parameter family of Hamiltonians \(H_{s,t}\) that satisfy
\[
(\phi^s)^*(dH_{s,t+1}) = dH_{s,t}
\]
and a compatible 2-form is given by
\[
\Omega = \omega + d(H_{s,t} dt).
\] (4.2.7)

We now examine the normalization condition. We compute
\[
\Omega^{n+1} = (n+1)\omega^n \wedge d(H_{s,t} dt) = (n+1) \frac{\partial H_{s,t}}{\partial s} \omega^n \wedge ds \wedge dt
\]
and its fiber integration
\[
\pi_!(\Omega^{n+1}) = (n+1) \left( \int_M \frac{\partial H_{s,t}}{\partial s} \omega^n \right) ds \wedge dt.
\]

Therefore the normalization condition \(\pi_!(\Omega^{n+1}) = 0\) is reduced to
\[
0 = \int_M \frac{\partial H_{s,t}}{\partial s} \omega^n = \frac{\partial}{\partial s} \left( \int_M H_{s,t} \omega^n \right).
\] (4.2.8)

In particular, if we consider the Hamiltonians satisfying the normalization condition \(\int_M H_{s,t} \omega^n = C\) for all \((s, t) \in [0, 1]^2\), where \(C\) is a fixed constant, e.g., \(C = 0\), then (4.2.7) gives the expression for the unique coupling form.

**Exercise 4.2.6** Suppose that the symplectic fibration \(E \to [0, 1] \times S^1\) with \(E|_{[0] \times S^1} \cong [0, 1] \times S^1\) is trivial on \([0, 1] \times S^1\) and equipped with a compatible
connection $\Gamma$ that is trivial at $\{0, 1\} \times S^1$. Prove that the fibration with such a connection is isomorphic with the mapping cylinder obtained from $\{\phi_{s,t}\}_{(s,t)\in[0,1]^2}$ with $\phi_{0,t} = id = \phi_{1,t}$ with the connection described in Example 4.2.5.

Finally we prove the following infinite-dimensional analog to Ambrose and Singer’s celebrated theorem about the relation between the holonomy and the curvature of the principal bundle (AS53): the curvature generates the holonomy group. Since Ambrose and Singer’s proof uses finite dimensionality in an essential way, we present a different coordinate-free proof directly using the simple version of a Campbell–Hausdorff-type formula. This proof applies to any fiber bundle whose structure group is a Fréchet Lie subgroup of the diffeomorphism group of the fiber, not just to the Hamiltonian diffeomorphism group.

**Proposition 4.2.7** Suppose $\Gamma$ satisfies (2) in Theorem 4.2.2. Let $b \in B$ and $v_1, v_2 \in T_b B$. Let $\eta_i$ for $i = 1, 2$ be any vector fields defined on a neighborhood $U$ of $b$ such that $\eta_i(b) = v_i$, and $\eta^\#_i$ their horizontal lifts. Then the vertical part of $[\eta^\#_1, \eta^\#_2]^v(b)$ defines a Hamiltonian vector field at the fiber $(E_b, \omega_b)$.

**Proof** We recall the basic formula for the Lie bracket of the vector field on a general manifold $M$:

$$\exp(tX)\exp(-tX)\exp(-tY)\exp\left(\frac{t^2}{2}[X,Y]\right) = \varphi(t), \quad (4.2.9)$$

where $\exp tX$ is the one-parameter subgroup of the vector field $X$ and

$$\text{dist}(\varphi(t), id) \leq Ct^3 \quad (4.2.10)$$

for some constant $C > 0$ depending only on $X$, $Y$ and $M$. (See (Spi79) for the proof.)

Let $v_1, v_2 \in T_b B$ and $\eta_1, \eta_2$ be given their local extensions on $B$ with $\eta_i(b) = v_i$. Then we have the curvature

$$\text{curv}(\eta_1, \eta_2) = [\eta^\#_1, \eta^\#_2] - [\eta_1, \eta_2]^\#.$$

We apply (4.2.9) to $\eta^\#_1, \eta^\#_2$ on $E$ and substitute $\sqrt{t}$ into $t$ to obtain

$$\exp(\sqrt{t}\eta^\#_1)\exp(\sqrt{t}\eta^\#_2)\exp(-\sqrt{t}\eta^\#_1)\exp(-\sqrt{t}\eta^\#_2)\exp\left(\frac{t}{2}[\eta^\#_1, \eta^\#_2]\right) = \varphi(\sqrt{t}).$$

Here we should point out that all the factors except the last one are $\sqrt{t}$-reparameterizations of the horizontal one-parameter subgroups. On multiplying by $\exp(-(t/2)[\eta^\#_1, \eta^\#_2])\exp((t/2)[\eta_1, \eta_2]^\#)$ on the right of the equation, we obtain
\[
\exp(\sqrt{\eta_1^\#})\exp(\sqrt{\eta_2^\#})\exp(-\sqrt{\eta_1^\#})\exp(-\sqrt{\eta_2^\#})\exp\left(\frac{t}{2} [\eta_1, \eta_2]^\#\right)
= \varphi(\sqrt{t})\exp\left(-\frac{t}{2} [\eta_1^\#, \eta_2^\#]\right)\exp\left(\frac{t}{2} [\eta_1, \eta_2]^\#\right).
\]

On the other hand, we have the general formula
\[
\exp(t(X + Y)) = \exp(tX)\exp(tY)(Id + O(t^2))
\]
and so
\[
\varphi(\sqrt{t})\exp\left(-\frac{t}{2} [\eta_1^\#, \eta_2^\#]\right)\exp\left(\frac{t}{2} [\eta_1, \eta_2]^\#\right)
= \varphi(\sqrt{t})\exp\left(\frac{t}{2} \left([-\eta_1^\#, \eta_2^\#] + [\eta_1, \eta_2]^\#\right)\right)\psi(t),
\]
where \(\psi(t)\) is an isotopy of diffeomorphism with \(\text{dist}(\psi(t), Id) = O(t^2)\).

Since \([-\eta_1^\#, \eta_2^\#] + [\eta_1, \eta_2]^\#\) is vertical, we have \(\pi \circ \exp\left((t/2) \left([-\eta_1^\#, \eta_2^\#] + [\eta_1, \eta_2]^\#\right)\right) = Id,\) so
\[
\pi \circ \varphi(\sqrt{t})\exp\left(\frac{t}{2} \left([-\eta_1^\#, \eta_2^\#] + [\eta_1, \eta_2]^\#\right)\right)\psi(t)) (b) = b + O(t^2).
\]
Therefore the projection curve
\[
t \mapsto \pi \left(\varphi(\sqrt{t})\exp\left(\frac{t}{2} \left([-\eta_1^\#, \eta_2^\#] + [\eta_1, \eta_2]^\#\right)\right)\psi(t)) (b) =: \alpha(t)\]
has its image within a distance \(O(\sqrt{t})\) from the point \(b \in B\) by (4.2.10). Now consider the one-parameter family of curves \(\alpha : [0, s] \to B\) for \(0 \leq s \leq \epsilon\) with \(\epsilon > 0\) sufficiently small. Since we have \(\text{dist}(b, \alpha(t)) \leq C \sqrt{t}\), we can complete them to a contractible piecewise-smooth loop issued at \(b\) by concatenating the short geodesic from \(\alpha(s)\) to \(b\) which has length \(\leq C \sqrt{s^3}\). It follows that the horizontal lift of this curve is precisely
\[
\exp(\sqrt{\eta_1^\#})\exp(\sqrt{\eta_2^\#})\exp(-\sqrt{\eta_1^\#})\exp(-\sqrt{\eta_2^\#})\exp\left(\frac{t}{2} [\eta_1, \eta_2]^\#\right)
\]
modulo the error of order \(O(\sqrt{t})\) for \(0 \leq t \leq s\). Since the holonomy map \(E_b \to E_b\) for such loops for each \(s\) is Hamiltonian, the generating vector field at \(t = 0\) of the associated Hamiltonian isotopy on \(E_b\) must be a Hamiltonian vector field. But a straightforward computation shows
\[
\left.\frac{d}{dt}\right|_{t=0} \exp(\sqrt{\eta_1^\#})\exp(\sqrt{\eta_2^\#})\exp(-\sqrt{\eta_1^\#})\exp(-\sqrt{\eta_2^\#})\exp\left(\frac{t}{2} [\eta_1, \eta_2]^\#\right)
= \frac{1}{2} \left([\eta_1^\#, \eta_2^\#] - [\eta_1, \eta_2]^\#\right).
\]
Therefore \([\eta_1^\#, \eta_2^\#] - [\eta_1, \eta_2]^\#\) is Hamiltonian, which is exactly what we want to show. This finishes the proof. \(\square\)
Now we are ready to give a proof of Theorem 4.2.2.

Proof of Theorem 4.2.2 First we prove that (1) implies (2). Let $\gamma$ be any contractible loop in $B$. Then the pull-back $\gamma^* E \to S^1$ becomes a symplectic fibration with the closed 2-form $\pi_2^* \Omega$, where $\pi_2 : \gamma^* E \to E$ is the canonical projection. By the classification of symplectic fibration over $S^1$ in Example 4.2.3, we may assume that $\gamma^* E \cong M_\phi$ for some symplectic diffeomorphism $\phi$.

Now let $w = w(s, t) : [0, 1] \times S^1 \to B$ be a contraction of $\gamma$ with

$$w|_{s=0} = \text{constant}, \quad w|_{s=1}(t) = \gamma(t).$$

Then consider the pull-back bundle $w^* E \to [0, 1] \times S^1$ and the pull-back form $\Omega_w := \pi_2^* \Omega$ by the map $\pi_2 : w^* E \to E$. Then the pair $(w^* E, \Omega_w)$ defines a fibration over $[0, 1] \times S^1$ induced by the closed 2-form $\Omega_w$. Now we compute the holonomy of $E$ over $\gamma$, which is the same as that of $w^* E$ along $\partial[0, 1] \times S^1$. By Example 4.2.5 and Exercise 4.2.6 therein, we can identify $w^* E$ with compatible 2-form $\Omega$ with a mapping cylinder with connection obtained by the explicit form

$$\Omega = \omega + d(H_s, dt),$$

where $H_{s,t}$ is the two-parameter family Hamiltonian satisfying $(\partial/\partial s) \int H_{s,t} \omega^n = 0$. Note that, since $w|_{s=0} = \text{constant}$, the normalization condition enables us to choose $H_{0,t} \equiv 0$. The normalization condition implies

$$\int_M H_{s,t} \omega^n = 0$$

for all $s \in [0, 1]$. Then the holonomy along $\gamma$ is nothing but the flow of the Hamilton equation $\dot{x} = X_{H_{s,t}}(x)$ and hence it is Hamiltonian.

Now we prove that (2) implies (1). We will write down the unique coupling form explicitly. Since any 2-form compatible with the connection $\Gamma$ at $E_b$ restricts to the fiber symplectic form, it suffices to specify the values of the pair $\Omega(\xi, \eta^\flat)$ and $\Omega(\eta^\flat_1, \eta^\flat_2)$, where $\xi$ is tangent to the fiber and $\eta^\flat$ and $\eta^\sharp$ are the horizontal lifts of tangent vectors $\eta$ and $\eta_i$ of $B$ at $b \in B$. First we set

$$\Omega(\xi, \eta^\flat) = 0$$

to achieve compatibility of $\Omega$ to the connection $\Gamma$. To specify $\Omega(\eta^\flat_1, \eta^\flat_2)$, we apply Proposition 4.2.7 and define

$$\Omega^e(\eta^\flat_1, \eta^\flat_2) := H_{\eta_1, \eta_2}(\pi(e)), \quad (4.2.11)$$

where $H_{\eta_1, \eta_2}$ is the fiberwise Hamiltonian function on $E_{\pi(e)}$ whose associated Hamiltonian vector field becomes the curvature vector $[\eta^\sharp_1, \eta^\sharp_2] = [\eta_1, \eta_2]^\sharp$. In other words, $H_{\eta_1, \eta_2}^b$ is a function on $E_b$ such that
\[ dH^b_{\eta_1,\eta_2} = [\eta_1^\# , \eta_2^\# ]^v_b. \] (4.2.12)

Now we evaluate the 2-form on \( B \) against \((v_1, v_2) \in T_b B\) and get

\[ \pi_!(\Omega^{n+1})(b)(v_1, v_2) = \int_{E_b} (v_1)^\# \wedge (v_2)^\# \Omega^{n+1}. \]

By definition, we have

\[ (v_1)^\# \wedge (v_2)^\# \Omega^{n+1} = (n + 1)\Omega(v_1^\# , v_2^\# )\Omega^n. \]

But we have \( \Omega(v_1^\# , v_2^\# ) = H_{v_1, v_2}(b) \), and hence the normalization condition \( \pi_!(\Omega^{n+1}) = 0 \) reduces to \( \int_{E_b} H_{v_1, v_2}(b)\omega^b = 0 \). Of course, with this normalization of the Hamiltonian, \( H_{v_1, v_2} \) is uniquely determined by the pair \((v_1, v_2) \) in \( T_b B \) and hence the coupling form \( \Omega_! \) is uniquely determined.

Finally we show the closedness of the form \( \Omega = \Omega_! \) defined. Since we already know the vertical closedness of \( \Omega \), it suffices to show that

\[ d\Omega(\xi, \eta_1^\# , \eta_2^\# ) = 0 \]

for \( \xi \) vertical and \( \eta_i^\# \) horizontal. But a straightforward computation shows that

\[
\begin{align*}
d\Omega(\xi, \eta_1^\# , \eta_2^\# ) &= \xi[\Omega(\eta_1^\# , \eta_2^\# )] + \eta_1^\# [\Omega(\eta_2^\# , \xi)] + \eta_2^\# [\Omega(\xi, \eta_1^\# )] \\
&\quad - \Omega([\xi, \eta_1^\# ], \eta_2^\# ) - \Omega([\eta_1^\# , \eta_2^\# ], \xi) - \Omega([\eta_2^\# , \xi], \eta_1^\# ).
\end{align*}
\]

It is easy to check the following:

\[
\begin{align*}
\xi[\Omega(\eta_1^\# , \eta_2^\# )] &= \xi[H_{\eta_1, \eta_2}], \\
\eta_1^\# [\Omega(\eta_2^\# , \xi)] &= \eta_2^\# [\Omega(\xi, \eta_1^\# )] = 0, \\
\Omega([\eta_1^\# , \eta_2^\# ], \xi) &= \Omega([\eta_1^\# , \eta_2^\# ], \xi) = dH_{\eta_1, \eta_2}(\xi).
\end{align*}
\]

For the study of the other terms, the following lemma is useful.

**Lemma 4.2.8** Let \( \xi \) be a vertical vector field, i.e., \( d\pi(\xi) = 0 \) and consider the horizontal lift \( \eta^\# \) of a vector field \( \eta \) on \( B \). Then the vector field \([\xi, \eta^\# ]\) is vertical.

**Proof** Note that both \( \xi \) and \( \eta^\# \) are projectable in that

\[ d\pi(\xi) = 0, \quad d\pi(\eta^\# ) = \eta; \]

i.e., they are \( \pi \)-related to the zero vector field and to \( \eta \) on \( B \), respectively. Therefore, we have
4.2 Hamiltonian fibration

\[ d\pi[\xi, \eta^\sharp] = [d\pi(\xi), d\pi(\eta^\sharp)] = [0, \eta] = 0. \]

Hence \([\xi, \eta^\sharp]\) is vertical. \(\square\)

This lemma then implies that

\[ \Omega([\xi, \eta^\sharp_1], \eta^\sharp_2) = \Omega([\eta^\sharp_2, \xi], \eta^\sharp_1) = 0. \]

Therefore we have obtained

\[ d\Omega(\xi, \eta^\sharp_1, \eta^\sharp_2) = \xi[H_{\eta^\sharp_1, \eta^\sharp_2}] - dH_{\eta^\sharp_1, \eta^\sharp_2}(\xi) = 0. \]

This proves the closedness of \(\Omega\). Since the last statement is easy to check, we leave its proof to readers. This finishes the proof of Theorem 4.2.2. \(\square\)

By construction, there is a close relationship between the coupling form \(\Omega_\Gamma\) and the curvature of the connection \(\Gamma\), which we summarize in the following. This is an immediate translation of (4.2.12).

**Theorem 4.2.9** Let \(E \to B\) be a Hamiltonian fibration and let \(\Gamma\) be a Hamiltonian connection. Denote

\[ Y_{\eta^\sharp_1, \eta^\sharp_2} = (\pi^\ast \text{curv}(\Gamma))(\eta^\sharp_1, \eta^\sharp_2), \]

\[ H_{\eta^\sharp_1, \eta^\sharp_2} = \Omega_\Gamma(\eta^\sharp_1, \eta^\sharp_2) \]

as a vector field and as a function on \(E\), respectively. Then we have

\[ Y_{\eta^\sharp_1, \eta^\sharp_2} \Omega = dH_{\eta^\sharp_1, \eta^\sharp_2} \quad (4.2.13) \]

for any \(\eta^\sharp_1, \eta^\sharp_2 \in TB\).

We now specialize the above discussion on the curvature to the case of the product fibration \(E = \Sigma \times (M, \omega)\) with \(\Sigma\) a compact two-dimensional surface, and relate it to the family of Hamiltonian flows. Regard the product \(E = \Sigma \times (M, \omega)\) as a bundle of a symplectic manifold whose structure group is \(\text{Symp}_0(M, \omega)\), the identity component of \(\text{Symp}(M, \omega)\).

Assuming that \(M\) is connected, we obtain the exact sequence

\[ 0 \to \mathbb{R} \to C^\infty(M) \to \text{ham}(M, \omega) \to 0, \quad (4.2.14) \]

where \(\text{ham}(M, \omega)\) is the set of Hamiltonian vector fields on \((M, \omega)\). This encodes that the correspondence between Hamiltonian functions and Hamiltonian vector fields is unique modulo the addition by constant functions. This exact sequence is an exact sequence of Lie algebras with a trivial bracket on
\( \mathbb{R} \), the Poisson bracket on \( C^\infty(M) \) and the Lie bracket on \( \text{ham}(M, \omega) \). In our convention, the second map is given by 
\[ h \mapsto -X_h. \]

When \( M \) is closed, this exact sequence splits by the map taking the mean 
\[ \int_M : C^\infty(M) \to \mathbb{R}; \ h \mapsto \int_M h \ d\mu. \]

Considering a family of Hamiltonians parameterized by a space \( \Sigma \), which induces a natural exact sequence
\[ 0 \to \Omega^1(\Sigma, \mathbb{R}) \to \Omega^1(\Sigma, C^\infty(M)) \to \Omega^1(\Sigma, \text{ham}(M, \omega)) \to 0, \]
we obtain the isomorphism
\[ \Omega^1(\Sigma, C^\infty_0(M)) \cong \Omega^1(\Sigma, \text{ham}(M, \omega)), \]
where we denote \( C^\infty_0(M) = \ker \int_M \).

Now let \( K \in \Omega^1(\Sigma, C^\infty(M)) \) and denote by \( P_K \) the corresponding one-form of \( \Omega^1(\Sigma, \text{ham}(M, \omega)) \). Then for each choice of \( \xi \in C^\infty(T\Sigma) \), \( K(\xi) \) gives a function on \( M \) and so a Hamiltonian vector field \( P_K(\xi) = X_{K(\xi)} \) on \( M \). One important quantity associated with the one-form \( K \) is a two-form on \( \Sigma \), denoted by \( R_K \): this is defined by
\[ R_K(\xi_1, \xi_2) = \xi_1[K(\xi_2)] - \xi_2[K(\xi_1)] - \{K(\xi_2), K(\xi_1)\} \quad (4.2.15) \]
for two vector fields \( \xi_1, \xi_2 \), where \( \xi_1[K(\xi_2)] \) denotes the directional derivative of the function \( K(\xi_2)(z, x) \) with respect to the vector field \( \xi_1 \) as a function on \( \Sigma \), holding the variable \( x \in M \) fixed. It follows from the expression that \( R_K \) is tensorial on \( \Sigma \). In fact, this is the curvature of the connection associated with \( K \).

**Exercise 4.2.10** (See (Ba78)) Consider the trivial bundle \([0, 1]^2 \times (M, \omega) \to [0, 1]^2 \). Prove that the connection \( P_K \) is flat, i.e., \( R_K = 0 \) if and only if \( P_K \) as an \((s, t)\)-dependent vector field can be integrated into the two-parameter family of Hamiltonian isotopies \( \Lambda : (s, t) \mapsto \Lambda(s, t) \in \text{Ham}(M, \omega) \).

### 4.3 Hamiltonian fibrations, connections and holonomies

Let \( S \) be a compact oriented connected surface of genus \( g \) with \( h \geq 1 \) boundary components: \( \partial S = \bigcup_{i=1}^h \partial_i S \), where \( \partial_i S \) is the \( i \)th component of \( \partial S \). The orientation on \( S \) in turn induces the boundary orientation on each \( \partial_i S \).
Consider a Hamiltonian fibration $E \to S$ with connection $\Gamma$, with its fiber modeled with $(M, \omega)$. Let $\omega_S$ be an area form on $S$ with $\int_S \omega_S = 1$, and let $\Omega$ be the coupling form on $E$ associated with $\Gamma$.

**Proposition 4.3.1** There exists a positive constant $\lambda_0 > 0$ such that the two-form $\Omega + \lambda \pi^* \omega_S$ is nondegenerate on $E$ for all $\lambda > \lambda_0$.

**Proof** We would like to show that there exists $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$ the map $\xi \mapsto \xi(\Omega + \lambda \pi^* \omega_S)$ is an isomorphism from $TE \to T^* E$, i.e., that, whenever $\xi$ satisfies $(\Omega + \lambda \pi^* \omega_S)(\xi, \eta) = 0$ for all $\eta \in T_e E$, then $\xi = 0$.

Suppose $\xi$ satisfies that $(\Omega + \lambda \pi^* \omega_S)(\xi, \eta) = 0$ for all $\eta \in T_e E$. Decomposing $\xi = \xi^v + \xi^h$, $\eta = \eta^v + \eta^h$ into the vertical and horizontal components, we obtain

$$0 = (\Omega + \lambda \pi^* \omega_S)(\xi^v + \xi^h, \eta^v + \eta^h)$$

$$= \Omega(\xi^v, \eta^v) + \Omega(\xi^h, \eta^h) + \lambda \pi^* \omega_S(\xi^h, \eta^h)$$

and so

$$\Omega(\xi^v, \eta^v) + \Omega(\xi^h, \eta^h) = -\lambda \pi^* \omega_S(\xi^h, \eta^h).$$

for all $\eta \in T_e E$. Here we use the definition of the connection $\Gamma$ compatible with $\Omega$, which implies $\Omega(T^v TM, T^h TM) = 0$.

By considering $\eta$ with $\eta^v = 0$ and $\eta^h = 0$, respectively, we obtain

$$\begin{cases} 
 \Omega(\xi^v, \eta^v) = 0, \\
 \Omega(\xi^h, \eta^h) = -\lambda \pi^* \omega_S(\xi^h, \eta^h)
 \end{cases}$$

for all $\eta$. By the fiberwise nondegeneracy of $\Omega$, we derive $\xi^v = 0$ from the first equation. On the other hand, the second equation is equivalent to

$$\Omega(\xi^h, \eta^h) = -\lambda \omega_S(d\pi \xi^h, d\pi \eta^h).$$

If $\xi^h = 0$, then $\xi = 0$, which finishes the proof. If not, the last equation implies

$$\frac{|\Omega(\xi^h, \eta^h)|}{|\omega_S(d\pi \xi^h, d\pi \eta^h)|} = \lambda$$

for any $\eta$ with $\omega_S(d\pi \xi^h, d\pi \eta^h) \neq 0$: Such an $\eta$ exists since $\omega_S$ is nondegenerate and $d\pi : \Lambda^2(HT E) \to \Lambda^2(TS)$ is an isomorphism.

Furthermore, since $S$ is compact, the quotient

$$\sup_{\eta, \omega_S(d\pi \xi^h, d\pi \eta^h) \neq 0} \frac{|\Omega(\xi^h, \eta^h)|}{|\omega_S(d\pi \xi^h, d\pi \eta^h)|} = \sup_{\xi, \eta, \omega_S(d\pi \xi^h, d\pi \eta^h) \neq 0} \frac{|\Omega(\xi^h, \eta^h)|}{|\omega_S(d\pi \xi^h, d\pi \eta^h)|}$$

for any $\xi^h = |\xi^h| = 1$. Such an $\eta$ exists since $\omega_S$ is nondegenerate and $d\pi : \Lambda^2(HT E) \to \Lambda^2(TS)$ is an isomorphism.
is not zero and finite. We denote this value by $\lambda_0$. Then we get a contradiction for any choice of $\lambda$ with $\lambda > \lambda_0$. Therefore we have proved that, whenever $\lambda > \lambda_0$, the map $\xi \mapsto \xi[(\Omega + \lambda \pi^* \omega_S)]$ is an isomorphism. This finishes the proof. \hfill \Box

We note that the symplectic forms $\Omega_{\Gamma} + \lambda \pi^* \omega_S$ are deformation-equivalent to one another for all $\lambda > \lambda_0$, and hence the family fixes a natural deformation class of symplectic forms on $E$ obtained by those

$$\omega_{\Gamma,\lambda} := \Omega_{\Gamma} + \lambda \pi^* \omega_S.$$  

### 4.3.1 Hamiltonian, connection and holonomy

Next we incorporate the holonomy effect into our discussion on the Hamiltonian fibration over the surface with boundary. We closely follow the exposition in (En00), restricting the discussion to the case of $G = \text{Ham}(M, \omega)$. Entov (En00) presents this discussion in the general context of $G$-bundles for an arbitrary Lie group $G$, not just for the Hamiltonian diffeomorphism group. Denote by $h$ the number of connected components of $\partial S$.

We note that the Hamiltonian connection $\Gamma$ determines a holonomy lying in $\text{Ham}(M, \omega)$ at each point $z \in \partial_i S$ in a trivialization. A different trivialization will give rise to an element conjugate in $\text{Ham}(M, \omega)$. Therefore, the holonomy at a given point in $\partial_i S$ determines a unique conjugacy class of $\text{Ham}(M, \omega)$ in $\text{Symp}(M, \omega)$.

Now let $C = (C_1, \ldots, C_h)$ be some conjugacy classes in $\text{Ham}(M, \omega)$ and denote by $\mathcal{L}(C)$ the set of Hamiltonian connections $\Gamma$ on $E \to S$ that are flat over a neighborhood of $\partial S$ and whose holonomy along $\partial_i S$ lies in $C_i$. By the flatness over a neighborhood of $\partial S$, the parallel transport of the fiber $E_{b_0}$, $b_0 \in \partial S$ trivializes $E$ in a neighborhood of $\partial_i S$ so that

$$E|_U \cong U \times E_{b_0}, \quad \Gamma_e : T_e(E)|_U = T_{\pi(e)} U \oplus E_{b_0}.$$  

Let $\Omega = \Omega_{\Gamma}$ be the coupling form of $\Gamma$. It fixes a natural deformation class of symplectic forms on $E$ obtained by those

$$\omega_{\Gamma,\lambda} := \Omega_{\Gamma} + \lambda \pi^* \omega_S,$$

where $\omega_S$ is an area form and $\lambda > 0$ is a sufficiently large constant. We will always normalize $\omega_S$ so that $\int_S \omega_S = 1$.

**Definition 4.3.2** The triple $(E, \Gamma, C)$ is a Hamiltonian fibration with connection $(E, \Gamma)$ and its holonomy $\phi_i$ along $\partial_i S$ lying in a conjugacy class contained in $C_i$. 


When \((g, h) = (0, 1), S = D^2\) and any Hamiltonian fibration \(E \to D^2\) is trivial. Let \((E, \Gamma, \mathcal{C})\) be a Hamiltonian fibration with connection and with holonomy along \(\partial D^2\). We choose a trivialization \(\Phi : E \to D^2 \times (M, \omega)\) and represent \(C = \{[\phi]\}\) with \(\phi \in \text{Ham}(M, \omega)\). Then we can represent the connection \(\Gamma|_{\partial D^2}\) by a \(1\)-periodic Hamiltonian \(H : S^1 \times M \to \mathbb{R}\) such that \([\phi_t^1] \in \mathcal{C}\) so that
\[
(T^h_e E) = \left\{ \left( a \frac{\partial}{\partial t}, \xi \right) \in T e \partial D^2 \times T_x M \mid e = (z, x) \in D^2 \times M, \xi = aX_H(x) \right\}.
\]
Or equivalently the connection 1-form \(A\) over \(\partial D^2\) can be written as
\[
A = X_H dt.
\]
We can extend \(H\) to a family \(D^2 \times M \to \mathbb{R}\), again denoted by \(H = H(z, x)\). We require the extension \(H\) to satisfy
\[
H(r, t, x) \equiv H(t, x)
\]
for all \(r \in (1 - \delta, 1]\) for some small \(\delta > 0\), which is made possible by the flatness of \(\Gamma\) in a neighborhood of \(\partial D^2\). More generally, for any \((g, h)\), we can choose a Hamiltonian connection \(\Gamma \in \mathcal{L}(\mathcal{C})\) on \(S\) so that the associated holonomies lie in the given conjugacy classes \(\mathcal{C} = (C_1, \ldots, C_h)\).

The following definition is introduced in \((\text{En}00)\).

**Definition 4.3.3** Consider a Hamiltonian fibration \(E \to S\). For a given Hamiltonian connection \(\Gamma\) of \(E\), we define size(\(\Gamma\)) by the supremum of \(1/\lambda_0\) for the \(\lambda_0\) appearing in Proposition 4.3.1 so that the form \(\Omega_\Gamma + \lambda \pi^* \omega_\Sigma\) is nondegenerate. We define the size
\[
\text{size}(E) := \sup_{\Gamma} \text{size}(\Gamma)
\]
over all Hamiltonian connections \(\Gamma\) of \(E\).

Proposition 4.3.1 implies that size(\(\Gamma\)) is always finite. We will estimate this size in terms of the \(K\)-area of the Hamiltonian fibration \(E \to \Sigma\) in the next section.

In the meantime, we want to relate the above discussion to a Hamiltonian fibration over a Riemann surface \((\Sigma, j)\) of genus \(g\) with \(k\) marked points \(\{z_1, \ldots, z_k\}\). The complex structure \(j\) induces the complex orientation on \(\Sigma\). We denote
\[
\hat{\Sigma} = \Sigma \setminus \{z_1, \ldots, z_k\}
\]
and fix analytic coordinates of the punctured neighborhood $D_i \setminus \{z_i\}$ of each puncture $z_i$ with either $[0, \infty) \times S^1$ or $(-\infty, 0] \times S^1$, with the standard complex structure on the cylinder so that each $\partial D_i$ is mapped to the boundary circle of the corresponding semi-cylinder. We call punctures of the first type incoming and those of the second type outgoing. In general we call a puncture with analytic coordinates a directed puncture.

We denote the identification by

$$\varphi^+_i : D_i \setminus \{z_i\} \to (-\infty, 0] \times S^1$$

for positive punctures and

$$\varphi^-_i : D_i \setminus \{z_i\} \to [0, \infty) \times S^1$$

for negative punctures. Assume that this identification extends to $D_i$ if we conformally identify $[0, \infty) \times S^1$ or $(-\infty, 0] \times S^1$ with $D^2 \setminus \{0\}$. We call such an identification the rational coordinates. We denote by $(\tau, t)$ the standard cylindrical coordinates on the cylinders. Then the rational coordinate maps $\varphi^\pm$ are uniquely determined up to translations by $\tau$.

We now define the real blow-up of $\Sigma$ at $\vec{z} = \{z_1, \ldots, z_k\}$ as a compact surface with a boundary whose components consist of $k$ circles that replaces each puncture $z_i$ by $\mathbb{R}P_1 \cong \mathbb{P}_{\mathbb{R}}(T_z \Sigma) \cong S^1$. We denote the corresponding topological space by

$$Bl_{\vec{z}}(\Sigma) = S.$$ 

We provide a natural conformal structure on $\text{Int}(Bl_{\vec{z}}(\Sigma))$ coming from the identification

$$\text{Int}(Bl_{\vec{z}}(\Sigma)) = \Sigma \setminus \{z_1, \ldots, z_k\}.$$ 

When we are also given analytic coordinates on $D_i \setminus \{z_i\}$, say $\varphi : D_i \setminus \{z_i\} \to [0, \infty) \times S^1$, we have a natural holomorphic embedding

$$[0, \infty) \times S^1 \to \text{Int}(Bl_{\vec{z}}(\Sigma))$$

mapping $(0, t) \in [0] \times S^1 \to \partial D_i$. This holomorphic embedding is uniquely determined up to the phase rotation on the domain.

When we prescribe holonomies along the boundary components of $Bl_{\vec{z}}(\Sigma)$, we call them the holonomies around the punctures of $\Sigma$.

**Lemma 4.3.4** Consider a Hamiltonian diffeomorphism $\phi$ and a Hamiltonian with $H \mapsto \phi$. Let $M_\phi \to S^1$ be the mapping circle of $\phi$ and consider the Hamiltonian connection on $M_\phi$ having the two-form

$$\omega + d(H dt)$$
as its coupling form. Consider any Hamiltonian fibration $E \rightarrow D^2$ with connection whose coupling form is given by $\Omega$ such that $\Phi_* \Omega$ restricts to the connection on $\partial S$ above. Consider any section $v : D^2 \rightarrow E$ that is horizontal along $\partial D^2$. Then, for any given trivialization $\Phi : E \rightarrow D^2 \times (M, \omega)$, we can write

$$\Phi \circ v(z) = (z, w_\Phi(z))$$

for a smooth map $w = w_\Phi : D^2 \rightarrow M$ such that $w|_{\partial D^2}(t) = \phi_H^t(x) := z^*_H(t)$ for some point $x \in M$. Furthermore, we have

$$-\int_{D^2} v^* \Omega = -\int_{D^2} w^* \omega - \int_{\partial D^2} H(t, z^*_H(t))dt. \quad (4.3.16)$$

**Proof** Let $(E, \Omega)$ be such a fibration with connection whose coupling form is $\Omega$. In a trivialization $\Phi : E \rightarrow D^2 \times (M, \omega)$, we can write $\Phi_* \Omega = \omega + \eta$, where $\eta$ is a closed form and $\eta(\xi, \cdot) \equiv 0$ for any $\xi \in TM$. We have that $\Omega$ also satisfies

$$(\Phi_* \Omega)|_{E|_{\partial S}} = \omega + d(H dt).$$

Consider a section $v : D^2 \rightarrow E$ given by $\Phi \circ v(z) = (z, w(z))$. Then we have

$$-\int_{D^2} v^* \Omega = -\int_{D^2} (\Phi \circ v)^*(\Phi_* \Omega) = -\int_{D^2} (\Phi \circ v)^*(\omega + \eta)$$

$$= -\int_{D^2} w^* \omega - \int_{D^2} (\Phi \circ v)^* \eta.$$

Here the two-form $(\Phi \circ u)^* \eta$ is a form on $D^2$ that is closed and restricted to $w^* d(H dt) = d(w^* H dt)$ on $\partial D^2$. Since $H^1(D^2, \partial D^2) = 0$, the form $(\Phi \circ v)^* \eta$ is exact and so

$$\int_{D^2} (\Phi \circ v)^* \eta = \int_{\partial D^2} w^* H dt = \int_{\partial D^2} H(t, w|_{\partial D^2}(t))dt.$$

On the other hand, the horizontality of the section $u$ implies that $(\partial w/\partial t)(t) = X_H(t, w(t))$ for all $t \in \partial D^2$. This implies that $w|_{\partial D^2}(t) = \phi_H^t(x)$ for some $x \in M$ that finishes the proof. \qed

### 4.3.2 Curvature and weak couplings

Let $\omega_S$ be an area form on $S$ with $\int_S \omega_S = 1$ and let $\Omega$ be the coupling form on $E$ associated with $\Gamma$. For any such $\Omega$, we know that

$$\omega_{\Gamma, \lambda} := \Omega_{\Gamma} + \lambda \pi^* \omega_S$$
becomes nondegenerate for all sufficiently large positive constant \( \lambda \). When we prescribe the holonomies along the boundary components, the choice of such a constant \( \lambda \) will depend on the given holonomies.

When we are given a trivialization \( \Phi : E \to S \times (M, \omega) \), we may identify a fiber with \((M, \omega)\). Then we may identify the curvature \( \text{curv}(\Gamma) \) of the connection \( \Gamma \) on \( \pi : E \to S \) with a two-form in \( \Omega^2(S; C_0^\infty(M)) \). Note that the Hamiltonian connection \( \Gamma \) determines a holonomy lying in \( \text{Ham}(M, \omega) \) at each point \( z \in \partial_i S \) in a trivialization.

For a given function \( h \in C^\infty(M) \), we denote
\[
\text{osc}(h) = \max_{x \in M} h(x) - \min_{x \in M} h(x).
\]
We recall that osc is invariant under the action of \( \text{Symp}(M, \omega) \) on \( C_0^\infty(M) \). Using this we can define the Hofer norm of the curvature \( \text{curv}(\Gamma) \in \Omega^2(S, C_0^\infty(M)) \) in the following way. For a given trivialization \( \Phi : E \to S \times M \), \( \text{curv}(\Gamma) \) induces a skew-symmetric bilinear form
\[
K_{\Gamma, \Phi} : T_z S \times T_z S \to C_0^\infty(M).
\]
Therefore we can define its Hofer norm by
\[
\|K_{\Gamma, \Phi}\| := \|K_{\Gamma, \Phi}(v, w)\| \quad \text{for } \omega_S(v, w) = 1.
\]
It is obvious that the right-hand side does not depend on the choice of \((v, w)\) with \( \omega_S(v, w) = 1 \).

**Exercise 4.3.5** Prove that the definition (4.3.17) does not depend on the choice of the trivialization. More specifically, consider two trivializations \( \Phi, \Psi \) and write
\[
\Psi \circ \Phi^{-1}(z, x) = (z, \phi_{\Psi \Phi}^z(x))
\]
for \( \phi_{\Psi \Phi}^z \in \text{Ham}(M) \). Then prove
\[
K_{\Gamma, \Psi}(v, w) = K_{\Gamma, \Phi}(v, w) \circ \phi_{\Psi \Phi}^z.
\]

Now we can define \( \|\text{curv}(\Gamma)\| = \|K_{\Gamma, \Phi}\| \) for a particular (and hence any) trivialization of \( E \to S \) and call it the curvature width of \( \Gamma \). Following (Gr96), (Po96) and (En00), we define the notion of the K-area of the connection \( \Gamma \) by
\[
K\text{-area}(\Gamma) = \frac{1}{\|\text{curv}(\Gamma)\|}
\]
and define the K-area of \( E \) by
\[
K\text{-area}(E) = \inf_\Gamma K\text{-area}(\Gamma).
\]
The following definition will be useful for the study of the triangle inequality later in Part 3 of this book.

**Definition 4.3.6** We define the $K$-area of $E$ equipped with the holonomy tuple $C = (C_1, \ldots, C_h)$ by

$$K\text{-area}(E; C) = \inf_{\Gamma \in \mathcal{L}(C)} K\text{-area}(\Gamma).$$

It follows that $K\text{-area}(E; C)$ does not depend on the order of the conjugacy classes $C_1, \ldots, C_\ell$.

Now we are ready to state a theorem relating the two invariants $\text{size}(E)$ and the $K\text{-area}(E)$ of $E$. Let $\Omega = \Omega_\Gamma$ be the coupling form of $\Gamma$. It fixes a natural deformation class of symplectic forms on $E$ obtained by those

$$\omega_{\Gamma, \lambda} := \Omega_\Gamma + \lambda \pi^* \omega_S,$$

where $\omega_S$ is an area form and $\lambda > 0$ is a sufficiently large constant. We will always normalize $\omega_S$ so that $\int_S \omega_S = 1$.

The following definition is an immediate generalization of Definition 4.3.3.

**Definition 4.3.7** Consider a Hamiltonian fibration $E \to S$ with a connection $\Gamma$ induced by a coupling form $\Omega$. Define $\text{size}(\Gamma)$ by the supremum of $1/\lambda$ appearing in Proposition 4.3.1 such that the form $\Omega_\Gamma + \lambda \pi^* \omega_S$ is nondegenerate. We define

$$\text{size}(E; C) = \sup_{\Gamma \in \mathcal{L}(C)} \text{size}(\Gamma).$$

The following theorem is equivalent to Theorem 3.7.4 (En00) proved by Entov.

**Theorem 4.3.8** Let $\pi : E \to \Sigma$ be a Hamiltonian fibration with connection $\Gamma \in \mathcal{L}(C)$. Then we have

$$\text{size}(E; C) \leq K\text{-area}(E; C).$$

**Proof** To check the nondegeneracy of $\Omega_\Gamma + \lambda \pi^* \omega_S$, it suffices to ensure that the form is nondegenerate on $HT_eE$ for all $e \in E$. But, for any given $\eta_1, \eta_2 \in T_{\pi(e)}S$, we have

$$\Omega(\eta_1^*, \eta_2^*) + \lambda \omega_S(\eta_1, \eta_2) = \text{curv}(\Gamma)(\eta_1, \eta_2) + \lambda \omega_S(\eta_1, \eta_2).$$
Therefore, if \( \text{curv}(\Gamma)(\eta_1, \eta_2) + \lambda \omega_\Sigma(\eta_1, \eta_2) = 0 \) for \( \omega_\Sigma(\eta_1, \eta_2) \neq 0 \), we have
\[
\lambda = -\frac{\text{curv}(\Gamma)(\eta_1, \eta_2)}{\omega_\Sigma(\eta_1, \eta_2)}.
\]
In particular, if \( \omega_\Sigma(\eta_1, \eta_2) = 1, \lambda = -\text{curv}(\Gamma)(\eta_1, \eta_2) \). This implies that, provided
\[
\lambda > \max_{z \in \Sigma} \max_{\eta_1, \eta_2 \omega_\Sigma(\eta_1, \eta_2) = 1} |\text{curv}(\Gamma)(\eta_1, \eta_2)|,
\]
the form \( \omega_{\Gamma, \lambda} = \Omega_\Gamma + \lambda \pi^* \omega_\Sigma \) is nondegenerate.

Equivalently, whenever we can find a Hamiltonian connection \( \Gamma \) such that
\[
\frac{1}{\lambda} < \max_{z \in \Sigma} \frac{1}{\|\text{curv}(\Gamma)(z)\|} = \frac{1}{\|\text{curv}(\Gamma)\|}
\]
the form \( \Omega_\Gamma + \lambda \pi^* \omega_\Sigma \) is nondegenerate. By taking the infimum of \( 1/\|\text{curv}(\Gamma)\| \) over all such \( \Gamma \), we have derived that, for any given \( 0 < \epsilon < \lambda \), the form \( \omega_{\Gamma, \lambda} \) is nondegenerate, provided that
\[
\frac{1}{\lambda} \leq K\text{-area}(E; C) + \epsilon.
\]
By taking the supremum of \( 1/\lambda \) over all such \( \lambda \) and then letting \( \epsilon \to 0 \), we have proved
\[
\text{size}(E; C) \leq K\text{-area}(E; C).
\]
This finishes the proof. \( \square \)

We can further refine our study of the \( \text{curv} \) and \( \text{size} \) of the fibration \( \pi : E \to \Sigma \) by varying the connection \( \Gamma \) among the connections which fix a homotopy class of holonomies.

### 4.3.3 Marked Hamiltonian fibration

Now let us consider a Hamiltonian fibration \( E \to S \) with its coupling form \( \Omega \) and denote by \( \nabla = \nabla_\Omega \) the associated symplectic connection. By definition, the holonomy of \( \nabla \) along a loop based at a point \( z \) defines an element \( \text{Ham}(E_z, \omega) \). In particular, when each boundary component of \( \partial S \) carries a marked point \( * \) and an identification
\[
(E_*, \Omega_z) \to (M, \omega)
\]
the holonomy defines an element in \( \text{Ham}(M, \omega) \).

**Definition 4.3.9** Let \( S \) be a compact Riemann surface with boundary. Denote \( \partial S = \bigsqcup_{i=1}^h \partial_i S \).
(1) A **marked Hamiltonian fibration** $E \to S$ is a Hamiltonian fibration $(E, \Omega)$ with a marked point $z_i \in \partial S$ with an identification $(E_{z_i}, \Omega_{z_i}) \cong (M, \omega)$.

(2) We say that a marked Hamiltonian fibration $E$ has boundary holonomy datum $(\phi_1, \ldots, \phi_h) \in \text{Ham}(M, \omega)^h$ if it satisfies

$$\text{hol}_\Gamma(\partial_i S; z_i) = \phi_i.$$  \hfill (4.3.21)

The following definition is also useful for the study of Hamiltonian geometry in terms of Floer theory later.

**Definition 4.3.10** Let $S$ be as above. A **marked Hamiltonian fibration with connection** is a Hamiltonian fibration $(E, \Omega)$ with trivializations

$$\Phi_{\partial S} : (E|_{\partial S}, \Omega|_{\partial S}) \to \partial S \times (M, \omega)$$

along the boundary $\partial S$. We say that a marked Hamiltonian fibration with connection has boundary holonomy datum $C = (C_{H_1}, \ldots, C_{H_h})$ if

$$\text{hol}_{\mathcal{V}, \partial}(\partial_i S) = \phi_{H_i}^i, \quad i = 1, \ldots, h.$$  \hfill (4.3.22)

We note that, if we choose a different trivialization,

$$\Phi'_{\partial S} : (E|_{\partial S}, \Omega|_{\partial S}) \to \partial S \times (M, \omega),$$

then the corresponding holonomies conjugate to each other and hence the conjugacy classes depend not on the trivializations but on the identification at the marked points in $\partial S$. 

5

Hofer’s geometry of Ham($M, \omega$)

In the middle of the 1980s, Gromov and Eliashberg (El87) proved the celebrated symplectic $C^0$-rigidity theorem, which states that any non-symplectic diffeomorphism cannot be the (locally) uniform limit of a sequence of symplectic diffeomorphisms. This was the first indication of the existence of the concept of symplectic topology. The main idea of Eliashberg’s proof relies on a construction of certain $C^0$-type invariants that are preserved under the symplectic diffeomorphisms. This construction of a $C^0$-type invariant was then completed by Gromov in his seminal paper (Gr85) as a consequence of Gromov’s non-squeezing theorem. Ekeland and Hofer (EkH89), (EkH90) formalized this type of $C^0$-invariant into the notion of symplectic capacity. We refer the reader to the book (HZ94) for a complete exposition of capacities on the classical phase space $\mathbb{R}^{2n}$. Hofer (H90) carried out a very canonical construction of a capacity on any symplectic manifold, the so-called displacement energy, exploiting the geometry of Hamiltonian flows. This construction is based on a remarkable (infinite-dimensional) bi-invariant Finsler geometry of the group Ham($M, \omega$) of Hamiltonian diffeomorphisms. The associated bi-invariant norm is called the Hofer norm on Ham($M, \omega$). The nondegeneracy of the norm was then established by Hofer (H90), Polterovich (Po93) and Lalonde and McDuff (LM95a) in increasing generality.

In this section, we assume that either $M$ is closed or $M$ is open, and, when $M$ is non-compact or has a boundary, all functions are compactly supported in the interior.

5.1 Normalization of Hamiltonians

We start by considering an arbitrary manifold $M$, which is not necessarily symplectic.
5.1 Normalization of Hamiltonians

We recall from the general Lie group theory that, on any Lie group $G$, any norm on the Lie algebra $\mathfrak{g}$ that is invariant under the adjoint action leads to a bi-invariant Finsler norm on the tangent bundle $TG$ via the right translation from the identity to the given group element of $G$.

In the case of $\text{Diff}(M)$, the Lie algebra corresponds to the set $\mathfrak{X}(M)$ of vector fields with the Lie bracket

$$[X, Y] = \mathcal{L}_X(Y).$$

We recall that $\mathcal{L}_X(Y)$ is defined by

$$\mathcal{L}_X Y = \frac{d}{dt} \bigg|_{t=0} (\phi_t)_* Y,$$

where $\phi_t$ is the flow of $\dot{x} = X(x)$. The adjoint action of $\text{Diff}(M)$ on $\mathfrak{X}(M)$ is realized by

$$\text{Ad}_\phi(Y) = \frac{d}{dt} \bigg|_{t=0} \phi_t \phi^{-1},$$

where $\psi_t$ is the one-parameter subgroup with $\psi_0 = id$ and $Y = (d/dt)|_{t=0} \psi_t$. In other words, $\text{Ad}_\phi(Y)$ is the vector field defined by

$$\text{Ad}_\phi(Y)(x) = d\phi(Y(\phi^{-1}(x))) = \phi_* (Y)(x), \quad (5.1.1)$$

which is precisely the push-forward of $Y$ under $\phi$.

Now we consider a symplectic manifold $(M, \omega)$. By considering a smooth one-parameter subgroup of $\text{Ham}(M, \omega)$, we have shown in Section 2.3 that the Lie algebra $\text{ham}(M, \omega)$ of $\text{Ham}(M, \omega)$ is canonically isomorphic to $\mathcal{C}^\infty(M)/\mathbb{R}$ with respect to the Poisson bracket associated with $\omega$: more precisely, the assignment

$$f \mapsto -X_f = X_{-f}$$

is a Lie algebra isomorphism between $(\mathcal{C}^\infty(M)/\mathbb{R}, \{\cdot, \cdot\})$ and $(\mathfrak{X}(M), [\cdot, \cdot])$. We recall the identity

$$\psi^*(X_f) = X_{f \circ \psi},$$

and

$$\{f \circ \psi, g \circ \psi\} = \{f, g\} \circ \psi$$

for any symplectic diffeomorphism $\psi$. Therefore, the pull-back $\psi^*X_f = (\psi^{-1})_* (X_f)$ corresponding to the adjoint action by $\psi^{-1}$ on $\mathcal{C}^\infty_0(M)$ is precisely given by the composition

$$f \circ \psi.$$
Obviously this action extends to the whole of the diffeomorphism group \( \text{Diff}(M) \).

For many reasons, as we will see later, it is useful to realize the quotient space \( C^\infty(M)/\mathbb{R} \) as a subspace of \( C^\infty(M) \) by choosing a good slice of the action of \( \mathbb{R} \), which is given by the addition of constants on \( C^\infty(M) \). There are two canonical choices of such slices that are invariant under the adjoint action of \( \text{Symp}(M,\omega) \). We will call any element in each of the slices normalized. We now explain the normalizations in detail, dividing our discussion into two cases: the closed case and the open case. The latter means that \( M \) is either non-compact or has a nonempty boundary.

First consider the case in which \((M,\omega)\) is closed. Let \( d\mu \) be the Liouville measure of \( M \) induced by the volume form \( \omega^n/n! \). By definition this measure is \( \text{Symp}(M,\omega) \)-invariant and finite on closed \( M \).

**Definition 5.1.1** We define

\[
C^\infty_0(M) = \{ f \in C^\infty(M) \mid \int_M f \, d\mu = 0 \}.
\]

We call any element in \( C^\infty_0(M) \) mean-normalized.

The following is an easy consequence of Liouville’s lemma, Lemma 2.3.2.

**Lemma 5.1.2** Assume \( M \) is closed. Then

\[
\int_M \{f,g\} \, d\mu = 0
\]

for all \( f, g \in C^\infty(M) \) and hence

\[
\{C^\infty(M),C^\infty(M)\} \subset C^\infty_0(M).
\]

In particular, \( C^\infty_0(M) \) is \( \text{Ham}(M,\omega) \)-invariant under the adjoint action.

**Proof** Using \( \mathcal{L}_{X_g} \omega = 0 \) and Cartan’s magic formula, we compute

\[
\int_M \{f,g\} \, d\mu = \int_M L_{X_g} (f) \, d\mu = \int_M L_{X_g} (f) \frac{\omega^n}{n!} = \int_M L_{X_g} \left( f \frac{\omega^n}{n!} \right) = \int_M d \left( X_g f \frac{\omega^n}{n!} \right) = 0.
\]

Next consider the case of \((M,\omega)\) open. The following normalization has been used in the literature.
Definition 5.1.3 Define

\[ C^\infty_c(M) = \{ f \in C^\infty(M) \mid \text{supp } f \text{ is compact and contained in } \text{Int}(M) \} \]

Here \( c \) stands for ‘compactly supported’.

However this normalization does not seem to be the optimal one from the point of view of Hamiltonian geometry, which we now elaborate.

On an open \((M, \omega)\), it is natural to consider compactly supported Hamiltonian diffeomorphisms (or, more generally, symplectic diffeomorphisms).

Definition 5.1.4 A compactly supported symplectic diffeomorphism \( \phi \) is called

1. a weakly compactly supported Hamiltonian diffeomorphism if there exists a Hamiltonian \( H : [0, 1] \times M \to \mathbb{R} \) such that its Hamiltonian vector field \( X_H \) is compactly supported and \( \phi = \phi^1_H \), or
2. a strongly compactly supported Hamiltonian diffeomorphism if there exists a compactly supported Hamiltonian \( H : [0, 1] \times M \to \mathbb{R} \) and \( \phi = \phi^1_H \).

We denote by \( \text{Ham}^{wc}(M, \omega) \) and \( \text{Ham}^{sc}(M, \omega) \), respectively, the corresponding sets of such diffeomorphisms.

Obviously we have the inclusions

\[ \text{Ham}^{sc}(M, \omega) \subset \text{Ham}^{wc}(M, \omega) \subset \text{Symp}^{c}(M, \omega) \]

Now we analyze the Lie algebra of each of these Lie groups. It is apparent that the Lie algebra of \( \text{Symp}^{c}_{0}(M, \omega) \) is the set of compactly supported locally Hamiltonian vector fields. Next we note that by definition we have

\[ \text{Ham}^{sc}(M, \omega) = \{ \phi \in \text{Symp}^{c}(M, \omega) \mid \phi = \phi^1_H, \text{supp}(H) \subset \text{Int}(M) \text{ is compact} \} \]

where \( \text{supp}(H) \) is defined by

\[ \text{supp}(H) = \bigcup_{t \in [0,1]} \text{supp}(H_t). \]

It is easy to see that the Lie algebra of \( \text{Ham}^{sc}(M, \omega) \) is given by

\[ \text{ham}^{sc}(M, \omega) = C^\infty_c(M). \]

Finally we consider \( \text{Ham}^{wc}(M, \omega) \). This requires some discussion on the end structure of \( M \). We recall that \( M \) is said to have a finite number of ends if there exists an increasing sequence of compact subsets.
such that $\bigcup_i M_i = M$ and there exists $N_0 \in \mathbb{Z}_+$ for which the number of connected components of $M \setminus M_i$ remains constant for all $i \geq N_0$. We call the constant the number of ends of $M$. It is easy to check that the number does not depend on the choice of exhaustion (5.1.2). We denote by $\infty_\alpha$ the end $\alpha$ of $M$.

**Definition 5.1.5** Suppose that $M$ has a finite number of ends. We define 

$$C_{ac}^\infty(M) = \left\{ f \in C^\infty(M) \left| \sum_{\alpha \in \text{End}(M)} f(\infty_\alpha) = 0 \right. \right\}.$$ 

One can easily check that, for any compactly supported Hamiltonian vector field $X$, we can always add a constant to its generating Hamiltonian so that the new Hamiltonian is contained in $C_{ac}^\infty(M)$. Therefore $C_{ac}^\infty(M)$ is canonically isomorphic to the Lie algebra of $\text{Ham}^{wc}(M, \omega)$. It follows that $C_{ac}^\infty(M)$ is $\text{Ham}^{wc}(M)$-invariant and 

$$\dim(C_{ac}^\infty(M)/C_c^\infty(M)) = \#\text{End}(M) - 1.$$ (5.1.3)

The following example shows that $\text{Ham}^{sc}(M, \omega) \subsetneq \text{Ham}^{wc}(M, \omega)$, i.e., there exists a weakly compactly supported Hamiltonian diffeomorphism that is not strongly compactly supported. It follows from (5.1.3) that, for such a diffeomorphism to exist, the number of ends of $M$ must be at least 2.

**Example 5.1.6** Consider the cylinder $M = \mathbb{R} \times S^1$ with the symplectic form $\omega = d\tau \wedge dt$. We note that the number of ends of $\mathbb{R} \times S^1$ is 2. Consider the Hamiltonian $f : \mathbb{R} \times S^1 \to \mathbb{R}$ defined by

$$f(\tau, t) = \rho(\tau),$$

where $\rho : \mathbb{R} \times [0, 1]$ is a monotone function such that

$$\rho(\tau) = \begin{cases} -\frac{1}{2}, & \text{for } \tau \leq -K, \\ \frac{1}{2}, & \text{for } \tau \geq K \end{cases}$$ (5.1.4)

for some $K > 0$. The corresponding Hamiltonian vector field $X_\rho$ is compactly supported and its flow $\phi^t_\rho$ lies in $\text{Ham}^{wc}(M, \omega)$.

**Exercise 5.1.7** Prove that $\phi^t_\rho$, $t \neq 0$ does not lie in $\text{Ham}^{sc}(M, \omega)$. 
5.2 Invariant norms on $C^\infty(M)$ and the Hofer length

There is a natural $\text{Diff}(M)$-invariant norm on $C^\infty(M)/\mathbb{R}$ defined by

$$\|h\| = \text{osc}(h) = \max h - \min h. \quad (5.2.5)$$

It turns out that the norm (5.2.5) is the unique $\text{Diff}(M)$-invariant norm if $n = \dim M \geq 2$ in the following sense (Che00).

**Theorem 5.2.1** (Chekanov) *Let $\dim M \geq 2$. Suppose that $\|\cdot\|'$ is any $\text{Diff}(M)$-invariant norm on $C^\infty_0(M)$. Then there exists $\lambda \geq 0$ such that $\|f\|' = \lambda \|f\|$ for all $f \in C^\infty_0(M)$.***

When we consider a symplectic manifold $(M, \omega)$, we can take the above-mentioned normalized slices to realize the quotient space $C^\infty(M)/\mathbb{R}$ as an invariant subset of $C^\infty(M)$ depending on whether $M$ is closed or open. Let $G$ be either of the two groups $\text{Ham}^{sc}(M, \omega)$ and $\text{Ham}^{wc}(M, \omega)$. We note that, when $M$ is closed, $\text{Ham}(M, \omega) = \text{Ham}^{sc}(M, \omega)$ since $\partial M = \emptyset$. However, we take $C^\infty_0(M)$ for the corresponding Lie algebra when $M$ is closed, whereas we take $C^\infty_c(M)$ when $M$ is open.

The following symplectic (or Hamiltonian) counterpart of Chekanov’s theorem above is an important question to ask.

**Question 5.2.2** *Let $(M, \omega)$. What is the classification of $G$-invariant norms of the corresponding Lie algebra $\mathfrak{g}$?*

A combination of the results obtained by Ostrover and Wagner (OW05) and by Buhovsky and Ostrover (BO13) provides a rather complete answer to this question.

**Theorem 5.2.3** (Ostrover–Wagner) *Assume $(M, \omega)$ is closed. Let $\|\cdot\|$ be a $\text{Ham}(M, \omega)$-invariant norm on $C^\infty_0(M)$ such that $\|\cdot\| \leq C\|\cdot\|_\infty$ for some constant $C$, but the two norms are not equivalent. Then the associated Finsler pseudo-distance on $\text{Ham}(M, \omega)$ vanishes identically.*

**Theorem 5.2.4** (Buhovsky–Ostrover) *For a closed symplectic manifold $(M, \omega)$, any $\text{Ham}(M, \omega)$-invariant pseudonorm $|\cdot|$ on $C^\infty_0(M)$ is dominated from above by the $L_\infty$ norm, i.e., $|\cdot| \leq C|\cdot|_\infty$ for some constant $C \geq 0$.***
The above invariant norm then induces a Finsler-type metric on the Lie group $G$ in the following way. We can identify the tangent space of $G$ at $\phi \in G$ with $\mathfrak{g}$ itself by the assignment
\[ f \mapsto f \circ \phi, \quad f \in C^\infty_0(M). \]
Considering $f$ as an element in the Lie algebra $\mathfrak{g}$, we may consider the assignment
\[ \phi \mapsto f \circ \phi \]
as the right-invariant vector field on $G$ generated by $f$.

To simplify our exposition, we will restrict the discussion to the case in which $M$ is closed. We will briefly indicate the differences for the open case later.

Suppose $\lambda : [0, 1] \to \text{Symp}(M, \omega)$ is a Hamiltonian path issued at the identity. Recall the notations
\[ \text{Tan}(\lambda)(t, x) := H(t, \phi^t_H(x)), \]
\[ \text{Dev}(\lambda)(t, x) := H(t, x) \]
from Definition 2.3.15.

Now we define the Finsler length of the path $\lambda$ on $\text{Ham}(M, \omega)$ with respect to the above-mentioned Finsler structure in the standard way.

**Definition 5.2.5** Let $\lambda \in \mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega))$ and let $H$ be the Hamiltonian generating the given Hamiltonian path $\lambda$, i.e., $\lambda(t) = \phi^t_H \circ \lambda(0)$. The *Hofer length* of the Hamiltonian path $\lambda$ is defined by
\[ \text{leng}(\phi_H) = \int_0^1 \left( \max_x H(t, \phi^t_H(x)) - \min_x H(t, \phi^t_H(x)) \right) dt. \] (5.2.6)

Obviously $\text{leng}(\phi_H)$ coincides with the familiar expression for the $L^{(1, \infty)}$ norm
\[ \|H\| = \int_0^1 \text{osc} \ H_t \, dt = \int_0^1 (\max_x H_t - \min_x H_t) \, dt. \]

The following simple exercises will be useful later for the calculus of the Hofer length function.

**Exercise 5.2.6** Let $H, K : [0, 1] \times M \to \mathbb{R}$. Prove the following.

1. $\text{leng}(\phi^t_H)^{-1} \phi_K = \text{leng}(\phi^{-1}_K \phi_H)$.
2. $\text{leng}(\phi_H \phi_K) \leq \text{leng}(\phi_H) + \text{leng}(\phi_K)$.
3. $\text{leng}(\phi_H) = \text{leng}(\phi^1_H)^{-1}$.

(Hint: The proofs are essentially translations of Proposition 2.3.8.)
The length function provides a natural topology on $\mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega), id)$ in the following way.

**Definition 5.2.7** Let $\lambda, \mu \in \mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega), id)$. We define a distance function on $\mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega), id)$ by

$$d(\lambda, \mu) := \text{leng}(\lambda^{-1} \circ \mu), \quad \lambda, \mu \in \mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega), id),$$

where $\lambda^{-1} \circ \mu$ is the Hamiltonian path $t \in [0, 1] \mapsto \lambda(t)^{-1}\mu(t)$. We call the induced topology the **Hofer topology** of $\mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega), id)$.

### 5.3 The Hofer topology of $\text{Ham}(M, \omega)$

The remarkable Hofer pseudonorm of Hamiltonian diffeomorphisms introduced in (H90), (H93) is defined by

$$\|\phi\| = \inf_{H \mapsto \phi} \|H\| = \inf_{H \mapsto \phi} \text{leng}(\phi_H).$$

The proof of the following proposition is an interesting exercise involving the Hamiltonian algebra.

**Proposition 5.3.1** Let $\phi, \psi \in \text{Ham}^c(M, \omega)$. Then the following hold.

1. **(Symmetry)** $\|\phi\| = \|\phi^{-1}\|$.
2. **(Triangle inequality)** $\|\phi\psi\| \leq \|\phi\| + \|\psi\|$.
3. **(Symplectic invariance)** $\|h\phi h^{-1}\| = \|\phi\|$ for all $h \in \text{Symp}(M, \omega)$.

**Proof** We recall from (2.3.22) the formula for the inverse Hamiltonian $\overline{H}(t, x) = -H(t, \phi_H^{-1}(x))$ generating $(\phi_H^{-1})^{-1}$. Obviously we have

$$\max_x \overline{H}_t = -\min_x H_t,$$

$$\min_x \overline{H}_t = -\max_x H_t.$$  

It then follows that $\|\overline{H}\| = \|H\|$. Furthermore, we know that taking ‘bar’ operation twice is the identity operation. On combining these we derive (1) by taking the infimum of $\|\overline{H}\| = \|H\|$ over all $H \mapsto \phi$.

For (2), we recall the composition Hamiltonian $H \# F$ defined by

$$H \# F(t, x) = H(t, x) + F(t, (\phi_H)^{-1}(x)),$$
which generates $\phi \psi$ if $H \mapsto \phi$ and $F \mapsto \psi$. The latter fact in particular implies

$$\|\phi \psi\| \leq \inf_{H \mapsto \phi, F \mapsto \psi} \|H \# F\|.$$

(5.3.9)

On the other hand, we have

$$\max_x (H \# F)_t \leq \max_x H_t + \max_x F_t,$$

$$-\min_x (H \# F)_t \leq -\min_x H_t - \min_x F_t.$$

Then we derive

$$\|H \# F\| = \int_0^1 \left( \max_x (H \# F)_t - \min_x (H \# F)_t \right) dt$$

$$\leq \int_0^1 (\max_x H_t + \max_x F_t - \min_x H_t - \min_x F_t) dt$$

$$= \|H\| + \|F\|.$$

(5.3.10)

On combining (5.3.9) and (5.3.10), we have proved (2).

For the proof of (3), we first recall that for a given Hamiltonian $H$ the pull-back $h^* H = H \circ h$ generates $h^{-1} \circ \phi'_H \circ h$ for any symplectic diffeomorphism $h : M \to M$. Obviously we have $\|H\| = \|H \circ h\| = \|h^* H\|$. On taking the infimum over all $H \mapsto \phi$, we have proved (3).

□

This norm then gives rise to a natural metric topology on $\text{Ham}(M, \omega)$, which we call the Hofer topology of $\text{Ham}(M, \omega)$. This is the original definition in (H90), (H93).

We now follow the exposition from (OhM07) to provide another more conceptual way of defining the Hofer topology on $\text{Ham}(M, \omega)$ starting from the Hofer topology on $\mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega), \text{id})$. We first rewrite the definition of $\text{Ham}(M, \omega)$. Denote by $\mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega), \text{id})$ the set of Hamiltonian paths issued at the identity, and by

$$\text{ev}_1 : \mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega), \text{id}) \to \text{Symp}(M, \omega); \quad \text{ev}_1(\lambda) = \lambda(1)$$

(5.3.11)

the evaluation map. Then it follows from the definition of $\text{Ham}(M, \omega)$ that we have

$$\text{Ham}(M, \omega) = \text{ev}_1(\mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega), \text{id})).$$

(5.3.12)

This expression provides a natural topology in a more manifest way.

**Proposition 5.3.2** The Hofer topology on $\text{Ham}(M, \omega)$ is equivalent to the strongest topology for which the evaluation map (5.3.11) is continuous.
5.4 Nondegeneracy and symplectic displacement energy

Proof By definition of the Hofer topology on $\mathrm{Ham}(M, \omega)$, it is easy to see that the map $\mathrm{ev}_{1}^{\mathrm{ham}}$ is continuous. Therefore the Hofer topology is weaker than the strongest topology. For the converse, it is enough to prove that an open set of $\mathrm{Ham}(M, \omega)$ in the strongest topology is also open in the Hofer topology.

Let $\mathcal{U} \subset \mathrm{Ham}(M, \omega)$ be open in the strongest topology and $\phi \in \mathcal{U} \subset \mathrm{Ham}(M, \omega)$. Let $H \mapsto \phi$. By the definition of the strongest topology, we know that $\mathrm{ev}_{1}^{-1}(\mathcal{U})$ is an open neighborhood of $\phi_{H}$ in $\mathcal{P}_{\mathrm{ham}}(\mathrm{Ham}(M, \omega), \mathrm{id})$. Therefore, recalling that

$$d(\phi', \phi) = \inf_{H'} \{d(\phi_{H'}, \phi_{H}) \mid H' \mapsto \phi', H \mapsto \phi\},$$

there exists $\epsilon > 0$ such that, whenever $d(\phi_{H'}, \phi_{H}) < 2\epsilon$, $\phi_{H'} \in \mathrm{ev}_{1}^{-1}(\mathcal{U})$, i.e., $\phi_{H'}^{1} \in \mathcal{U}$. Now we claim that, if $d_{\text{Hofer}}(\phi', \phi) < \epsilon$, then $\phi' \in \mathcal{U}$, which will show that $\mathcal{U}$ is open in the Hofer topology. By definition, we can find $F$ such that $\phi_{F}^{1} = \phi'$ and

$$d(\phi_{F}, \phi_{H}) < 2\epsilon,$$

which shows that $\phi_{F} \in \mathrm{ev}_{1}^{-1}(\mathcal{U})$ and in particular we have $\phi' \in \mathcal{U}$. This finishes the proof. □

Now a big question is whether this topology is trivial or not. We will discuss this question in the next section.

5.4 Nondegeneracy and symplectic displacement energy

In this section, we first explain the idea of the proof of nondegeneracy of the pseudonorm which was employed by Hofer (H90).

Here is the key concept that Hofer uses in the proof of nondegeneracy on $\mathbb{C}^{n}$.

Definition 5.4.1 (Displacement energy) Let $A \subset M$ be any subset of $M$. We define

$$e(A) = \inf_{H} \{||H|| \mid \overline{A} \cap \phi_{H}^{1}(A) = \emptyset\}$$

and call $e(A)$ the displacement energy of $A$.

It is manifest that the displacement energy is invariant under the action of $\mathrm{Symp}(M, \omega)$.

Suppose that $\phi : M \rightarrow M$ is not the identity. Then there is a point $x \in M$ such that $\phi(x) \neq x$. By continuity of $\phi$, we can indeed find an open neighborhood $U$ of $x$ such that
\[ \phi(U) \cap \overline{U} = \emptyset. \]

By the Darboux theorem, we may choose \( \overline{U} \) such that it is the image of a symplectic embedding

\[ g : B^{2n}(r) \subset \mathbb{C}^n \to (M, \omega); \quad g(0) = x. \]

In general, the image of any such symplectic embedding is called a *symplectic ball* of Gromov area \( \lambda = \pi r^2 \). We denote by \( B(\lambda) \) any such symplectic ball of Gromov area \( \lambda \).

We note that the set of ‘null’ Hamiltonian diffeomorphisms \( \phi \) with \( \|\phi\| = 0 \) forms a normal subgroup. Therefore Banyaga’s theorem implies that the set of null Hamiltonian diffeomorphisms is either \( \{id\} \) or the full group \( \text{Ham}(M, \omega) \). Therefore we need only show that \( e(B(\lambda)) > 0 \) for a symplectic ball \( B(\lambda) \subset M \) with \( \lambda > 0 \), for this will then imply that there exists a Hamiltonian diffeomorphism \( \text{id} \neq \phi \) with \( \|\phi\| \geq e(B(\lambda)) > 0 \) by the definition of displacement energy.

To demonstrate non-triviality of the proof of the positivity \( e(B(\lambda)) > 0 \), we state a proposition that shows that it takes ‘zero energy’ to move a point to any other point in \( M \).

**Proposition 5.4.2** Let \( x_0, x_1 \) be any two points in \( M \). Then, for any given \( \delta > 0 \), there exists \( H \) such that \( \phi^1_H(x_0) = x_1 \) and \( \|H\| \leq \delta \). In other words, the displacement energy of a point is zero.

**Proof** We first start with the case \( M = \mathbb{R}^{2n} \) with the canonical symplectic form. Noting that the symplectic group \( Sp(2n, \mathbb{R}) \) acts transitively on \( \mathbb{R}^{2n} \), we may assume that \( x_0 = 0 \), the origin, and \( x_1 = (s, 0, \ldots, 0) \) for some \( s \in \mathbb{R} \).

Next we consider an autonomous Hamiltonian \( h = sp_1 \). Then \( X_{sp_1} = s \partial/\partial q^1 \) and its flow is given by \( \phi^t_{sp_1}(x) = x + t(x, \ldots, 0) \) for \( x \in \mathbb{R}^{2n} \). In particular, we have \( \phi^1_{h}(x_0) = x_1 \). Note that \( p_1(\phi^t_{h}(x_0)) \equiv 0 \) for all \( t \).

Finally we consider the line segment \( I := \overline{x_0x_1} \) and denote its \( \delta \)-neighborhood by \( N_\delta(I) \). Choose a cut-off function \( \rho : \mathbb{R}^{2n} \to \mathbb{R} \) satisfying

\[ \rho(x) = \begin{cases} 1, & \text{for } x \in N_{\delta/3}(I), \\ 0, & \text{for } x \notin N_{\delta/2}(I), \end{cases} \quad 0 \leq \rho(x) \leq 1. \]

Then we choose the Hamiltonian \( H = \rho \cdot h \). Since \( H \equiv h \) on \( N_{\delta/2}(I) \subset I \), we have \( \phi^1_{H}(x_0) = \phi^1_{h}(0) = x_1 \). Furthermore, it follows from the construction of \( H \) that
5.4 Nondegeneracy and symplectic displacement energy

\[ \max H \leq \frac{\delta}{2}, \quad \min H \geq -\frac{\delta}{2} \]

and hence \( ||H|| \leq \delta \). Since \( \delta > 0 \) can be chosen arbitrarily small, this finishes the proof for the case of \( \mathbb{R}^{2n} \).

Now we consider the general case of \((M, \omega)\). We choose any embedded curve \( \alpha : [0, 1] \to M \) with \( \alpha(0) = x_0, \alpha(1) = x_1 \). Choose a finite partition

\[ 0 = t_0 < t_1 < \cdots < t_N = 1 \]

with its mesh \( 1/N \). By choosing a sufficiently large \( N \), we can choose a finite covering of \( \alpha([0, 1]) \) by symplectic balls \( g_i : B^{2n}(r) \to M, i = 0, \ldots, N - 1 \) of uniform size so that each \( \alpha([t_i, t_{i+1}]) \) is contained in one of the balls in the covering. It suffices to prove that it takes ‘zero energy’ to move the point \( \alpha(t_i) \) to \( \alpha(t_{i+1}) \) for each \( i = 0, \ldots, N - 1 \). However, this follows from the result on \( \mathbb{R}^{2n} \) in the first part of the proof: we apply the first part on \( B^{2n}(r) \subset \mathbb{R}^{2n} \) and then push forward the corresponding Hamiltonian after multiplying it by a suitable cut-off function. This will finish the proof. \qed

Exercise 5.4.3 Complete the details of the last part of the argument of the proof above. More precisely, prove that one can choose a covering of \( \alpha([0, 1]) \) by a finite number of symplectic balls of uniform size, and give the details of transferring the result on \( B^{2n}(r) \) to \((M, \omega)\) via \( g_i \).

On the other hand, the following non-trivial theorem shows that the displacement energy of a symplectic ball \( B(\lambda) \) for any \( \lambda > 0 \) is positive.

Theorem 5.4.4 Let \( B(\lambda) \) be any symplectic ball \( \lambda > 0 \).

1. (Hofer (H90)) Let \( B^{2n}(r) \) be the standard ball in \( \mathbb{R}^{2n} \). Then \( e(B^{2n}(r)) = \pi r^2 \).
2. (Lalonde–McDuff (LM95a)) Let \( B(\lambda) \subset M \) be any symplectic ball of Gromov area \( \lambda > 0 \), \( e(B(\lambda)) \geq \lambda/2 \).

This theorem is sometimes called the energy–capacity inequality. The following optimal version of the energy–capacity inequality was proved by Usher (Ush10a). We will give its proof in Section 22.3.

Theorem 5.4.5 (Optimal energy–capacity inequality, (Ush10a)) Let \((M, \omega)\) be an arbitrary symplectic manifold and let \( B(\lambda) \subset M \) be a symplectic ball. Then

\[ e(B(\lambda)) \geq \lambda. \]
An immediate corollary of the positivity result of $e(B(\lambda)) > 0$ for $\lambda > 0$ follows.

**Corollary 5.4.6** The Hofer norm on $\text{Ham}(M, \omega)$ is nondegenerate, i.e., $\|\phi\| = 0$ if and only if $\phi = \text{id}$.

We would like to mention that the Hofer distance $d(\phi, \phi') := \|\phi^{-1}\phi'\|$ is intrinsic in that it is nothing but the distance of the Finsler metric induced by the $Ad$-invariant norm $\|h\| = \text{osc}(h) = \max h - \min h$ defined on $C^\infty_0(M)$. Obviously this Hofer norm on $\text{Ham}(M, \omega)$ is bi-invariant and continuous with respect to $C^\infty$ topology of $\text{Ham}(M, \omega) \subset \text{Diff}(M)$. In this regard, Buhovsky and Ostrover proved the following uniqueness result combining Theorem 5.2.4 and Theorem 5.2.3.

**Theorem 5.4.7** (Buhovsky–Ostrover) Let $(M, \omega)$ be closed. Then any bi-invariant Finsler pseudo-metric on $\text{Ham}(M, \omega)$, obtained by a pseudonorm $| \cdot |$ on $C^\infty_0(M)$ that is continuous in the $C^\infty$ topology, is either identically zero or equivalent to Hofer’s Finsler metric. In particular, any nondegenerate bi-invariant Finsler metric is equivalent to the Hofer metric.

Before closing this section, we would like to re-examine the scheme of the nondegeneracy proof outlined at the beginning of this subsection. It is easy to see from the proof that one can equally use any collection $C$ of subsets of $M$, besides the collection of symplectic balls $B(\lambda)$, as long as the collection satisfies the following two properties.

- For any open subset $U \subset M$, there is an element $A \in C$ such that $A \subset U$.
- For any $A \in C$, one can prove $e(A) > 0$.

It was Polterovich (Po93) who observed that the set of Lagrangian submanifolds (or even that of Lagrangian tori) satisfies these two properties. Usage of Lagrangian submanifolds suits very well the method of Gromov’s pseudoholomorphic curves. We will present a simple proof given in (Oh97c) of the positivity of the displacement energy of any compact Lagrangian submanifold in tame symplectic manifolds as an easy application of the study of Floer’s perturbed Cauchy–Riemann equation. This positivity property was first proved by Polterovich (Po93) for rational Lagrangian submanifolds and by Chekanov (Che98) for general Lagrangian submanifolds. Polterovich used Gromov’s figure-8 trick (Gr85) in his proof and Chekanov used a rather sophisticated Floer homology theory in the proof.
5.5 Hofer’s geodesics on Ham(M, ω)

Now let φ, ψ ∈ Ham(M, ω) and consider the Hofer distance d(φ, ψ). Using the invariance of the Hofer distance, without loss of any generality, we may assume ψ = id. In the point of view of the Hofer Finsler geometry of Ham(M, ω), it is natural to ask whether the infimum ∥φ∥ = inf_H{∥H∥ | H ↦ φ} is realized, i.e., whether there exists a Hamiltonian H such that ∥φ∥ = ∥H∥.

Definition 5.5.1 We say that two Hamiltonians F and G are equivalent and denote F ~ G if the associated Hamiltonian paths φ_F and φ_G are path-homotopic relative to the boundary t = 0, 1. Equivalently, F ~ G if the two Hamiltonians are connected by a one-parameter family of Hamiltonians {F^s}0≤s≤1 such that F^s ↦ φ for all s ∈ [0, 1]. We denote by [F] or ˜φ_F the equivalence class of F. Then we denote by Ham(M, ω) the set of such equivalence classes of Ham(M, ω).

Definition 5.5.2 Let φ_H : t ∈ [0, 1] ↦ φ_H^t be a Hamiltonian path with φ_H^0 = φ ∈ Ham(M, ω).

1) φ_H (or H) is called length minimizing if for any Hamiltonian path φ_H’ with

φ_H’^0 = φ_H^0, \hspace{1cm} φ_H’^1 = φ_H^1

we have leng(φ_H’) ≥ leng(φ_H), or equivalently, ∥H’∥ ≥ ∥H∥.

2) φ_H is called length minimizing in its homotopy class if the same holds for H’ that satisfies H ~ H’ in addition.

Hofer (H93) proved that the path of any compactly supported autonomous Hamiltonian on C^n is length minimizing as long as the corresponding Hamilton’s equation has no non-constant periodic orbit of period less than or equal to one.

One can define the notion of geodesics on Ham(M, ω) similarly as in Riemannian geometry. However, due to the nonsmoothness of the Hofer length function, giving a precise definition of geodesics is not a totally trivial matter. A precise variational definition of geodesics on Ham(M, ω) is given in (Po01), to which we also refer readers for some general discussions on geodesics.

We just quote a characterization of geodesics given by Bialy and Polterovich (BP94) in terms of the dynamical properties of the corresponding Hamiltonian.
Definition 5.5.3 A Hamiltonian $H$ is called quasi-autonomous if there exist two points $x_-, x_+ \in M$ such that

$$H(t, x_-) = \min_x H(t, x), \quad H(t, x_+) = \max_x H(t, x)$$

for all $t \in [0, 1]$.

In (BP94) and (LM95b), Bialy and Polterovich and Lalonde and McDuff proved that any length-minimizing (respectively, locally length-minimizing) Hamiltonian path is generated by quasi-autonomous (respectively, locally quasi-autonomous) Hamiltonian paths.

We now recall the Ustilovsky–Lalonde–McDuff necessary condition on the stability of geodesics.

Definition 5.5.4 Let $H : [0, 1] \times M \to \mathbb{R}$ be a Hamiltonian that is not necessarily time-periodic, and let $\phi^t_H$ be its Hamiltonian flow.

1. We call a point $p \in M$ a time $T$-periodic point if $\phi^T_H(p) = p$. We call $t \in [0, T] \mapsto \phi^t_H(p)$ a contractible time $T$-periodic orbit if it is contractible.

2. When $H$ has a fixed critical point $p$ over $t \in [0, T]$, we call $p$ over-twisted as a time $T$-periodic orbit if its linearized flow $d\phi^t_H(p); t \in [0, T]$ on $T_p M$ has a closed trajectory of period less than or equal to $T$.

Ustilovsky (Ust96) and Lalonde and McDuff (LM95b) proved that, for a generic $\phi$, in the sense that all its fixed points are isolated, any stable geodesic $\phi_t, 0 \leq t \leq 1$, from the identity to $\phi$ must have at least two fixed points that are under-twisted.

Hofer’s theorem on the length-minimizing property of the relevant autonomous paths has been generalized in (En00), (MSl01) and (Oh02)–(Oh05b) under the additional hypothesis that the linearized flow at each fixed point is not over-twisted, i.e., has no closed trajectory of period less than one. The following conjecture was raised by Polterovich, as Conjecture 12.6.D in (Po01) (see also (LM95b) and (MSl01)), and proved by Schlenk (Sch06) and Usher (Ush10a) in its final form.

Theorem 5.5.5 (Minimality conjecture) Any autonomous Hamiltonian path that has no contractible periodic orbits of period less than or equal to one is Hofer-length minimizing in its path-homotopy class relative to the boundary.

In Part 4 of this book, we will relate the length-minimizing property of quasi-autonomous paths to some homological property of the associated Floer
homology via some general construction of spectral invariants, and provide a simple criterion for the length-minimizing property in terms of the spectral invariants. Using this criterion, we will provide the proof of the minimality conjecture in Part 4 following the scheme used in (Oh02), (Oh05b) and (Ush10a).

On the other hand, not every pair of points in \( \text{Ham}(M, \omega) \) can be connected by a length-minimizing geodesic, as the following theorem by Ostrover (Os03) shows.

**Theorem 5.5.6** (Ostrover)  Let \((M, \omega)\) be a symplectic manifold with \(\pi_2(M) = \{0\}\). Then there exists an element \(\phi \in \text{Ham}(M, \omega)\) that cannot be connected by a length-minimizing geodesic to the identity.

A result of this kind on \(S^2\) was previously obtained by Lalonde and McDuff (LM95b).
Motivated by Gromov and Eliashberg’s rigidity theorem (El87), it is natural to define a symplectic homeomorphism (abbreviated as a sympleomorphism) as the $C^0$-limit of any sequence of symplectic diffeomorphisms inside the homeomorphism group. In (OhM07), the authors formulated the $C^0$-analog of the group of Hamiltonian diffeomorphisms and the notion of topological Hamiltonian flows. It turns out that the set of Hamiltonian homeomorphisms (abbreviated as hameomorphisms) forms a normal subgroup of the group of symplectic homeomorphisms, and it is believed to be a proper subgroup of the group of symplectic homeomorphisms. If this were true, it would provide the negative answer to an important question in dynamical systems, namely that of whether the area-preserving homeomorphism group of $S^2$ (or $D^2$) is simple or not.

This provides an important direction of study in symplectic topology, which one hopes would lead to the correct notion of $C^0$-symplectic topological space. In this chapter, we explore this largely uncharted territory of symplectic topology and topological Hamiltonian dynamics. We first explain the proof of a $C^0$-symplectic rigidity theorem closely following the proof of Ekeland and Hofer (EkH90) and then explain the definition of continuous Hamiltonian flows introduced by Müller and the author in (OhM07) and (Oh10).

### 6.1 $C^0$ symplectic rigidity theorem

We start by stating the celebrated Eliashberg–Gromov $C^0$-symplectic rigidity theorem.

**Theorem 6.1.1** (Eliashberg–Gromov) The subgroup $\text{Symp}(M, \omega)$ of symplectic diffeomorphisms is $C^0$-closed in $\text{Diff}(M)$. More precisely, if a sequence...
6.1 $C^0$ symplectic rigidity theorem

$\phi_j$ of symplectic diffeomorphism has a $C^0$-limit $\phi_\infty$ that is differentiable, then $\phi_\infty$ is a $(C^1)$-symplectic diffeomorphism.

Eliashberg’s proof (El87) relies on a structure theorem on the wave fronts of certain Legendrian submanifolds. The complete details of the proof of this structure theorem have not appeared in the literature. However, Gromov’s non-squeezing theorem can replace Eliashberg’s argument to give another proof. In fact, the existence of any symplectic capacity function on the set of open sets in $\mathbb{C}^n$ will provide a relatively straightforward proof of using Eliashberg’s argument in (El87).

In the remaining part of this section, we give the definition of one such symplectic capacity on $\mathbb{R}^{2n}$ using the non-squeezing theorem. We refer readers to the book (HZ94) for a nice exposition on the symplectic rigidity theory from the point of view of Hamiltonian dynamics and the classical critical-point theory.

Let $F \subset (M, \omega)$ be a closed subset.

**Definition 6.1.2** (Gromov area) For any closed subset $F$ of $M$, we define

$$c(F, \omega) = \sup\{\pi r^2 \mid \exists \text{ a symplectic embedding } \phi : (B^{2n}(r), \omega_0) \to (F, \omega)\}$$

and call it the Gromov area of $F$.

Obviously if $F$ has an empty interior, $c(F, \omega) = 0$.

We will restrict this definition to the subsets in $\mathbb{R}^{2n}$ for the rest of the section. The non-triviality of the above definition is one of the cornerstones of modern symplectic topology, whose proof is an immediate consequence of Gromov’s celebrated non-squeezing theorem. We refer readers to (Gr85) and also to Section 11.1 for its proof.

Next we define another capacity that is more useful in the proof of the rigidity theorem:

$$\overline{c}(F) = \inf\{\pi r^2 \mid \exists \psi \in \text{Symp}(\mathbb{R}^{2n}) \text{ with } \psi(F) \subset Z^{2n}(r)\}.$$ 

Obviously we have $\overline{c}(F) \geq c(F)$.

**Proposition 6.1.3** Let $c$ be any of $c$ and $\overline{c}$ on $\mathbb{C}^n$. Then $c$ has the following properties.

1. Monotonicity: $c(S) \leq c(T)$ if $S \subset T$.
2. Conformality: $c(\phi(F)) = |\alpha|c(F)$ if $\phi^*\omega_0 = \alpha \omega_0$ for $\alpha \neq 0$.
3. Normalization: $c(B^{2n}(1), \omega_0) = \pi = c(Z^{2n}(1), \omega_0)$. 

Proof. We only give the proof for $c$ and leave the other to readers. The monotonicity and conformality are easy consequences of the definition of $c$. Therefore we give the proof of the normalization axiom. First note that the monotonicity axiom implies $c(Z^{2n}(1), \omega_0) \geq c(B^{2n}(1), \omega_0)$. On the other hand, the non-squeezing theorem implies $c(Z^{2n}(1)) < c(B^{2n}(r)) = \pi r^2$ for all $r > 1$. Therefore we also have $c(Z^{2n}(1), \omega_0) \leq \pi$. Hence we have proved $c(Z^{2n}(1), \omega_0) = \pi$. The identity $c(B^{2n}(1), \omega_0) = \pi$ is trivial. □

Definition 6.1.4 (Symplectic capacity in $\mathbb{C}^n$) Any function $c$ defined on the subsets of $\mathbb{C}^n$ that satisfies the three properties stated in Proposition 6.1.3 is called a symplectic capacity function on $\mathbb{C}^n$.

Proposition 6.1.3 provides two examples of symplectic capacity on $\mathbb{C}^n$. The following proposition is an immediate consequence of the definition.

Proposition 6.1.5 For any symplectic capacity $c$ on $\mathbb{C}^n$, we have

$$c(F) \leq c(F) \leq \bar{c}(F)$$

for all subsets $F \subset \mathbb{C}^n$.

Proof. By the normalization axiom, it is easy to prove

$$\bar{c}(S) = c(S) = c(S) \quad (6.1.1)$$

for any symplectic ball. Let $F \subset \mathbb{C}^n$ be a bounded closed subset. By the definition of $\bar{c}$, there exists a symplectic diffeomorphism $\psi_1 : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\psi_1(F) \subset Z^{2n}(r + \epsilon)$$

with $\pi r^2 = \bar{c}(F)$. Therefore we derive

$$c(F) = c(\psi_1(F)) \leq \pi(r + \epsilon)^2.$$

Since this holds for all $\epsilon > 0$, we obtain $c(F) \leq \pi r^2 = \bar{c}(F)$.

Next we prove $c(F) \leq c(F)$. If $c(F) = 0$, there is nothing to prove and so we can assume $c(F) > 0$. By the definition of $c(F)$, for any given $0 < \epsilon < c(F)$, we can find a symplectic diffeomorphism $\psi_2$ such that

$$\psi_2(B^{2n}(r - \epsilon)) \subset F$$

where $\pi r^2 = c(F)$. By the symplectic invariance and the monotonicity, we obtain

$$c(B^{2n}(r - \epsilon)) = c(\psi_2(B^{2n}(r - \epsilon))) \leq c(F).$$
On the other hand, the normalization axiom of $c$ implies
\[ c(B^{2n}(r - \epsilon)) = \pi(r - \epsilon)^2 \]
and hence $\pi(r - \epsilon)^2 \leq c(F)$. Since this holds for any $0 \leq \epsilon < r$, we obtain $\pi r^2 \leq c(F)$ and so $c(F) \leq c(F)$. This finishes the proof. \hfill \Box

We will use the formal properties stated in this proposition to prove the Eliashberg–Gromov rigidity theorem in the rest of this section. We closely follow the scheme of the proof given by Ekeland and Hofer in (EkH90).

### 6.1.1 Properties of capacity-preserving linear maps

It turns out that the study of capacities of bounded ellipsoids and *coisotropic cylinders* under the linear maps plays an important role in the rigidity proof in (EkH90).

An (open) bounded ellipsoid $E = E_q$ is the subset defined by
\[ E_q = \{ z \in \mathbb{C}^n \mid q(z) < 1 \} \]
for a positive-definite quadratic form $q$ on $\mathbb{C}^n$. As an easy consequence of the normalization axiom, we have the following.

**Proposition 6.1.6** Let $c$ be any capacity. Then we have
\[ c(L \cdot E) = c(L \cdot E) = c(L \cdot E) \]
for any bounded ellipsoid $E$ and for any linear map $L$.

**Exercise 6.1.7** Prove this proposition.

**Definition 6.1.8** Let $W \subset \mathbb{C}^n$ be a coisotropic subspace and let $\Omega$ be a bounded open subset. We call the sum $\Omega + W$ a *coisotropic cylinder over $\Omega$*.

Note that, since the cylinder contains a small ball, we have $c(\Omega + W) > 0$. The following interesting lemma was proved by Ekeland and Hofer (EkH90). (Ekeland and Hofer phrased this hypothesis as $W^{\omega} null$. Here we use the more standard terminology *coisotropic* instead.)

**Lemma 6.1.9** (Lemma 3 (EkH90)) Let $W$ be a subspace of codimension 2 in $\mathbb{C}^n$. Consider the cylinder $\Omega + W$ for a bounded open subset $\Omega$. Then $c(\Omega + W) = \infty$ if and only if $W$ is coisotropic.

**Proof** We start with the following simple exercise.
Exercise 6.1.10 Let $W$ be a coisotropic subspace of codimension two. Then there exists a linear symplectic map $A \in Sp(2n, \mathbb{R})$ such that

$$A \cdot W = \{ z \in \mathbb{C}^n \mid \text{Re} \, z_1 = \text{Re} \, z_2 = 0 \}. $$

(See \cite{OhP05} for a complete description of the set of coisotropic subspaces of any codimension.)

Denote $z_j = q_j + \sqrt{-1} p_j$. After applying a linear symplectic coordinate change, we can take $\epsilon > 0$ so small that

$$z \in \Omega + W \quad \text{whenever} \quad q_1^2 + q_2^2 \leq \epsilon^2.$$ 

Such $\epsilon > 0$ exists by the normal form given in the above exercise. By definition of $\Omega + W$, we have

$$E_{q\epsilon, N} \subset \Omega + W$$

for all the ellipsoids $E_{q\epsilon, N}$ with $N > 0$, where the quadratic form $q_{\epsilon, N}$ is given by

$$q_{\epsilon, N} = \left( \frac{1}{\epsilon^2} q_1^2 + \frac{1}{N^2} p_1^2 \right) + \left( \frac{1}{\epsilon^2} q_2^2 + \frac{1}{N^2} p_2^2 \right) + \frac{1}{N^2} \sum_{k=3}^{2n} |z_k|^2.$$

But this quadratic form is conjugate to

$$\tilde{q}_{\epsilon, N} = \frac{1}{\epsilon N} (|z_1|^2 + |z_2|^2) + \frac{1}{N^2} \sum_{k=3}^{2n} |z_k|^2$$

under a symplectic coordinate change. Therefore we have

$$c(E_{q\epsilon, N}) = c(E_{\tilde{q}\epsilon, N}).$$

On the other hand, if $N > \epsilon$ so that $\epsilon N < N^2$, we have $c(E_{\tilde{q}\epsilon, N}) = \pi \epsilon N$. (Prove this!) Using the monotonicity of $c$, we derive $c(\Omega + W) \geq \pi \epsilon N$. Since this holds for all $N > 0$, we have proved $c(\Omega + W) = \infty$.

Next we prove the converse. Considering its contrapositive, we need to prove that, when $W$ is not coisotropic, we have $c(\Omega + W) < \infty$. Again we start with the following exercise.

Exercise 6.1.11 Suppose $W$ is a subspace of codimension 2 that is not coisotropic. Prove that there is a linear symplectic map $L$ such that

$$L \cdot W = \{ z \in \mathbb{C}^n \mid z_1 = 0 \}. $$

By this exercise, we may assume $W = \{ z \in \mathbb{C}^n \mid z_1 = 0 \}$ without loss of any generalities. Then, since $\Omega$ is bounded, there exists a sufficiently large $N > 0$ such that

$$\Omega + W \subset \{ z \in \mathbb{C}^n \mid |z_1|^2 < N^2 \} = Z^{2n}(N).$$
Then monotonicity implies \( c(\Omega + W) \leq c(\mathcal{Z}^{2n}(N)) = \pi N^2 < \infty \), which finishes the proof. \( \square \)

Next we prove the following lemma.

**Lemma 6.1.12**  Let \( \Psi \) be a linear map. Then \( \Psi \) is invertible, if it preserves a capacity \( c \) of any bounded ellipsoid.

**Proof**  Suppose \( c(\Psi(S)) = c(S) \) for any bounded ellipsoid \( S \).

It suffices to prove \( \Psi \) is surjective. Suppose to the contrary that \( \Psi(\mathbb{C}^n) \neq \mathbb{C}^n \). Then \( \Psi(\mathbb{C}^n) \) is contained in a hypersurface \( V \) through the origin. Applying a linear symplectic coordinate change if necessary, we may assume

\[
V = \{ z \in \mathbb{C}^n \mid \text{Re} z_1 = 0 \}.
\]

Consider any bounded ellipsoid \( S \), e.g., \( S = B^{2n}(r) \). Then \( \Psi(S) \) is a bounded subset contained in \( V \). Therefore there exists a sufficiently large \( N > 0 \) such that

\[
\Psi(S) \subseteq \left\{ z \left| \frac{1}{\epsilon^2} q_1^2 + \frac{1}{N^2} p_1^2 + \frac{1}{N^2} \sum_{k=2}^{n} |z_k|^2 \right\} \text{ for all } \epsilon > 0 .
\]

Then monotonicity implies

\[
c(\Psi(S)) \leq \pi \epsilon N
\]

for all \( \epsilon > 0 \) and hence \( c(\Psi(S)) = 0 \). On the other hand, we have \( 0 < c(S) = c(\Psi(S)) \), where the inequality follows from the normalization axiom and the equality from the symplectic invariance. Hence we obtain a contradiction and so \( \Psi \) must be invertible. \( \square \)

Next we recall the definition of the *symplectic dual* \( \Psi^* \) for a linear map. \( \Psi^* \) is the linear map satisfying

\[
\omega_0(\Psi(v), w) = \omega_0(v, \Psi^*(w)).
\]

An immediate corollary of the proposition and the lemma is the following.

**Corollary 6.1.13**  Let \( \Psi \) be a linear map preserving a capacity \( c \). Then \( \Psi \) maps any coisotropic subspace of codimension 2 to a coisotropic subspace of codimension 2. Equivalently \( (\Psi^*)^{-1} \) maps isotropic 2-planes to isotropic 2-planes.
Proof Assume that \( W \) is a coisotropic subspace of codimension 2. Let \( \Omega \) be a nonempty bounded open subset and consider the cylinder \( \Omega + W \). Proposition 6.1.9 then implies that \( c(\Omega + W) = \infty \). Since \( \Psi \) preserves \( c \), we obtain

\[
\infty = c(\Psi(\Omega + W)) = c(\Psi(\Omega) + \Psi(W)).
\]

The invertibility of \( \Psi \) implies that \( \Psi(\Omega) \) is again a nonempty bounded open subset and \( \Psi(W) \) is a subspace of codimension 2. Then Proposition 6.1.9 implies that \( \Psi(W) \) is coisotropic. This proves the first statement.

For the second statement, let \( U \) be an isotropic subspace of dimension 2 and denote \( W = U^{\omega_0} \), which is a coisotropic subspace of codimension 2. We recall the duality formula

\[
\Psi(U^{\omega_0}) = \left( (\Psi^*)^{-1} U \right)^{\omega_0}
\]

for any subspace \( U \subset \mathbb{C}^n \) and a linear map \( \Psi \). If \( U \) is an isotropic subspace of dimension 2, \( U^{\omega_0} \) is coisotropic of codimension 2 and so \( \Psi(U^{\omega_0}) \) is coisotropic by the first statement. Then, by the above duality formula, \( \left( (\Psi^*)^{-1} U \right)^{\omega_0} \) is coisotropic, or equivalently \( (\Psi^*)^{-1}(U) \) is isotropic. This finishes the proof. \( \square \)

Finally we prove the following general lemma.

**Lemma 6.1.14** Suppose \( (V, \omega) \) to be any symplectic vector space, and \( \Phi : V \to V \) is an invertible linear map that maps isotropic 2-planes to isotropic 2-planes. Then \( \Phi \) satisfies

\[
\Phi^* \omega = d \cdot \omega
\]

for some \( d \neq 0 \).

Proof Choose a Darboux basis

\[
\{e_1, \ldots, e_n, f_1, \ldots, f_n\}
\]

so that

\[
\omega(e_i, e_j) = 0 = \omega(f_i, f_j), \quad \omega(e_i, f_j) = \delta_{ij}.
\]

Then, since \( \Phi \) preserves isotropic 2-planes, we obtain

\[
\omega(\Phi(e_i), \Phi(e_j)) = 0 = \omega(\Phi(f_i), \Phi(f_j))
\]

and

\[
\omega(\Phi(e_i), \Phi(f_j)) = 0 \quad \text{for } i \neq j.
\]
Then, by invertibility of $\Phi$ and nondegeneracy of $\omega$, we obtain

$$\omega(\Phi(e_j), \Phi(f_j)) = d_j$$

for some $d_j \neq 0$. On the other hand, we also have

$$\omega(e_i - e_j, f_i + f_j) = 0$$

by simple calculation, i.e., the span $\{e_i - e_j, f_i + f_j\}$ is isotropic, and so is the span of $\{\Phi(e_i) - \Phi(e_j), \Phi(f_i) + \Phi(f_j)\}$. Therefore we obtain

$$0 = \omega(\Phi(e_i) - \Phi(e_j), \Phi(f_i) + \Phi(f_j))$$

$$= \omega(\Phi(e_i), \Phi(f_i)) - \omega(\Phi(e_j), \Phi(f_j)) = d_i - d_j$$

for all $i, j$. Denote by $d$ the common number $d_j$. Then we have proved

$$\omega(\Phi(e_j), \Phi(f_j)) = d \neq 0$$

for all $j$. On combining this with (6.1.2) and (6.1.3), we have proved that $\omega(\Phi(v), \Phi(w)) = d \cdot \omega(v, w)$ for all $v, w \in \mathbb{C}^n$, i.e., $\Phi^* \omega = d \cdot \omega$ as stated. \(\square\)

Now we are ready to finish the proof of the following theorem.

**Theorem 6.1.15** Suppose that $\Psi : \mathbb{C}^n \to \mathbb{C}^n$ is a capacity-preserving linear map. Then $\Psi$ is either symplectic or anti-symplectic.

**Proof** Denote $\Phi = (\Psi^*)^{-1}$. Then it follows from Corollary 6.1.13 and Lemma 6.1.14 that

$$\Phi^* \omega_0 = d \cdot \omega_0.$$ 

Then it follows that either $(1/\sqrt{|d|})\Phi$ or $(1/\sqrt{|d|})\Gamma \circ \Phi$ is symplectic, depending on the sign of $d$, where $\Gamma$ is any anti-symplectic isomorphism of $\mathbb{C}^n$. We note that $\Gamma$ is also capacity-preserving. Then $\sqrt{|d|}\Psi = ((1/\sqrt{|d|})\Phi^*)^{-1}$ or $\sqrt{|d|}(\Gamma^*)^{-1} \circ \Psi$. We note that $(\Gamma^*)^{-1}$ is also capacity-preserving. Therefore we have

$$c(\sqrt{|d|}\Psi(S)) = c(S)$$

for the first case. On the other hand, the conformality axiom implies that $c(\sqrt{|d|}\Psi(S)) = |d|c(\Psi(S))$ and so $|d| = 1$. A similar argument applies to the second case and hence the proof of $\Psi^* \omega_0 = \pm \omega_0$. \(\square\)
6.1.2 Proof of the rigidity theorem

In this section, we deal with general nonlinear capacity-preserving maps and their uniform limits. We will first prove the following theorem.

**Theorem 6.1.16**  Let \( \Phi_k \) be a sequence of continuous maps of the unit ball \( B^{2n}(1) \) into \( \mathbb{C}^n \) converging uniformly to \( \Phi \). Assume that all the \( \Phi_k \) preserve the capacity

\[
c(\Phi_k(S)) = c(S) \quad \text{for all } k
\]

of all bounded ellipsoids \( S \subset B^{2n}(1) \). If \( \Phi \) is differentiable at \( 0 \in B^{2n}(1) \), then the derivative \( d\Phi(0) \) is either symplectic or anti-symplectic.

Before giving the proof of this theorem, we state a couple of immediate corollaries of this theorem.

**Corollary 6.1.17**  Assume \( \Phi_k : B^{2n}(1) \to \mathbb{C}^n \) is a sequence of symplectic embeddings converging uniformly to a map \( \Phi : B^{2n}(1) \to \mathbb{C}^n \), which is differentiable at \( 0 \). Then \( d\Phi(0) \in Sp(2n\mathbb{R}) \).

**Proof**  Since symplectic embeddings preserve any capacity, Theorem 6.1.16 implies that \( d\Psi(0) \) is either symplectic or anti-symplectic. It remains to prove that \( d\Psi(0) \) is indeed symplectic.

Note that \( \Phi_k \) are all orientation-preserving maps. Let \( y_k = \Phi_k(0) \) and consider the map \( x \mapsto \tilde{\Phi}_k(x) := \Phi_k(x) - y_k \). Since \( \Phi_k \) uniformly converges, \( y_k \to y_\infty \) as \( k \to \infty \). Therefore the map \( \tilde{\Phi}_k \) uniformly converges \( \tilde{\Phi} \) given by

\[
\tilde{\Phi}(x) := \Phi(x) - y_\infty.
\]

Obviously we have \( d\Phi(0) = d\tilde{\Phi}(0) \) and so we can consider \( \tilde{\Phi}_k \) instead. This allows us to assume that \( \Phi_k(0) = 0 \) without any loss of generality, which we will do from now on.

Consider a small ball \( B^{2n}(\delta) \). Then, since \( \Phi_k(0) = 0 \) and \( \Phi_k \) are orientation-preserving, the winding number is

\[
\deg(\Phi_k|_{\partial B^{2n}(\delta)}; 0) = 1.
\]

Since the winding number is invariant under a uniform limit, we also have

\[
\deg(\Phi|_{\partial B^{2n}(\delta)}; 0) = 1.
\]

Since \( \Phi \) is assumed to be differentiable at 0, we have

\[
\deg(\Phi|_{\partial B^{2n}(\delta)}; 0) = \text{sign det } d\Phi(0).
\]
Hence we have proved that \( \text{sign} \det d\Phi(0) = 1 \) and so \( d\Phi(0) \) is orientation-preserving.

At this stage, if \( n \) is odd, \( d\Phi(0) \) must be symplectic, because an anti-symplectic map is orientation-reversing in these dimensions. When \( n \) is even, we consider the sequence of the product embeddings \( \text{Id}_C \times \Phi_k \) whose domains are \( C \times B^{2n}(1) \). It converges to \( \text{Id}_C \times \Phi \). Obviously each \( \text{Id}_C \times \Phi_k \) defines a symplectic embedding of \( B^{2(n+1)}(1) \) for which \( n+1 \) is odd. Furthermore, the limit \( \text{Id}_C \times \Phi \) has the derivative \( \text{Id}_C \times d\Phi(0) \) at 0. Therefore we conclude that \( \text{Id}_C \times d\Phi(0) \) is symplectic and so is \( d\Phi(0) \).

\[ \square \]

An immediate consequence of this corollary is the rigidity result that we have wanted to prove.

**Theorem 6.1.18**  
Let \((M, \omega)\) be any symplectic manifold. Then \( \text{Symp}(M, \omega) \) is closed in compact open \( C^0 \) topology in \( \text{Diff}(M) \).

**Proof**  
Using Darboux charts, we localize and apply Corollary 6.1.17. \[ \square \]

Now we give the proof of Theorem 6.1.16.

**Proof of Theorem 6.1.16**  
We will prove that the assumption in the theorem for any capacity \( c \) gives rise to the identity

\[ \bar{c}(d\Phi(0)(S)) = \bar{c}(S) \]  \hspace{1cm} (6.1.4)

for any bounded ellipsoids \( S \). Then we apply the linear rigidity result to \( d\Phi(0) \) to finish the proof. Hence it remains to prove (6.1.4).

Let \( S \) be a bounded ellipsoid such that \( \bar{S} \subset B^{2n}(1) \). By the definition of \( \bar{c} \), there exists a symplectic diffeomorphism \( \Psi \) such that \( \Psi^{-1}(\Phi(S)) \rightarrow Z^{2n}(r + \epsilon) \), where \( \pi r^2 = \bar{c}(\Phi(S)) \). Then we can find a bounded ellipsoid \( \tilde{S} \subset Z^{2n}(r + \epsilon) \) containing \( \Psi^{-1}(\Phi(S)) \). Therefore we have \( \Psi(\tilde{S}) \supset \Phi(S) \) and

\[ \bar{c}(\Psi(\tilde{S})) \leq \pi(r + \epsilon)^2 \]

by the monotonicity of \( \bar{c} \) and the normalization \( \bar{c}(Z^{2n}(r + \epsilon)) = \pi(r + \epsilon)^2 \). Since \( \Phi_k \rightarrow \Phi \) uniformly over \( B^{2n}(1) \supset S \), it follows that

\[ \Phi_k(S) \subset \Psi(\tilde{S}) \]

for all sufficiently large \( k \). The monotonicity of \( \bar{c} \) and (6.1.1) give rise to

\[ \bar{c}(S) = c(S) = c(\Phi_k(S)) \]
\[ \leq c(\Phi_k(S)) \leq c(\Psi(\tilde{S})) = \bar{c}(S) \]
\[ \leq \pi(r + \epsilon)^2. \]
Since this holds for all $\epsilon > 0$, we have proved
\[ c(S) \leq \pi r^2 = c(\Phi(S)) \]
for all bounded ellipsoids.

Now we consider the Alexander isotopy $\Phi_t : B^{2n}(1) \to \mathbb{C}^n$ for $t \in (0, 1)$
given by
\[ \Phi_t(x) = \frac{1}{t} \Phi(tx). \]
We note that $t \cdot S$ is also a bounded ellipsoid contained in $B^{2n}(1)$ and hence
\[ c(tS) \leq c(\Phi(tS)). \]
On dividing by $t$, we obtain
\[ c(S) \leq \frac{1}{t} c(\Phi(tS)) = c\left(\frac{1}{t} \Phi(tS)\right) = c(\Phi_t(S)) \quad (6.1.5) \]
for all $t \in (0, 1)$.

**Lemma 6.1.19** We have
\[ \lim_{t \to 0} c(\Phi_t(S)) \leq c(d\Phi(0)(S)). \]

**Proof** We note that for each fixed $t \in (0, 1)$ the map $\Phi_{k,t} \to \Phi_t$ uniformly on $B^{2n}(1)$ as $k \to \infty$. Choose a sequence $t_k \to 0$ and consider the diagonal subsequence $\Phi_{k,t_k}$. This sequence converges uniformly to $d\Phi(0)$ and preserves capacity $c$ for bounded ellipsoids. Therefore we can apply the same argument as in the proof of (6.1.5) to conclude that
\[ c(\Phi_{k,t_k}(S)) \leq c(d\Phi(0)(S)). \]
On the other hand, since $\Phi$ is differentiable at $0$, $\Phi_t$ uniformly converges to $d\Phi(0)$ on $B^{2n}(1)$. Since both $\Phi_{t_k}$ and $\Phi_{k,t_k}$ converge to $d\Phi(0)$ uniformly, it follows that
\[ \lim_{k \to \infty} c(\Phi_{t_k}) = \lim_{k \to \infty} c(\Phi_{k,t_k}). \]
This convergence and the above inequality finish the proof. \(\square\)

Combining this lemma with (6.1.5), we have proved that
\[ c(S) \leq c(d\Phi(0)(S)). \quad (6.1.6) \]
By the same argument as in the proof of Lemma 6.1.12 we derive from this that $d\Phi(0)$ is invertible.
It now remains to prove the opposite inequality. Let $\gamma > 0$ be given and consider $\tau > 0$ such that 

$$(1 + \gamma)\tau < 1.$$ 

**Lemma 6.1.20**  
For a sufficiently small $\tau > 0$,

$$\Phi_k((1 + \gamma)\tau S) \supset d\Phi(0)(\tau S)$$

for all sufficiently large $k$.

**Proof**  
We first recall that $\Phi_k(0) = 0 = \Phi(0)$. Since $\Phi$ is differentiable at $0$, there exists a continuous function $\epsilon : (0, 1) \rightarrow (0, \infty)$ such that $\epsilon(s) \rightarrow 0$ as $s \rightarrow 0$ and

$$|\Phi(x) - d\Phi(0)x| \leq \epsilon(|x|)|x|.$$ 

Furthermore, since $\Phi_k \rightarrow \Phi$ uniformly, for any given $\delta \in (0, 1)$ there exists $k(\delta)$ such that, for all $k \geq k(\delta)$,

$$|\Phi_k(x) - d\Phi(0)x| \leq \epsilon(|x|)|x| + \delta. \quad (6.1.7)$$

We set

$$\delta = \epsilon((1 + \gamma)\tau)(1 + \gamma)\tau.$$ 

Consider the homotopy

$$t \in [0, 1] \mapsto h_t := d\Phi(0) + t(\Phi_k - d\Phi(0))$$

between $d\Phi(0)$ and $\Phi_k$. Since $d\Phi(0)$ is invertible, there exists $d > 0$ such that

$$|d\Phi(0)x| \geq d|x|$$

for all $x$. On the other hand, we derive

$$|\Phi_k((1 + \gamma)\tau x) - d\Phi(0)(1 + \gamma)\tau x| \leq \epsilon((1 + \gamma)\tau|x|)(1 + \gamma)\tau|x| + \epsilon((1 + \gamma)\tau)(1 + \gamma)\tau$$

from (6.1.7) for all $x \in S$, provided that $\tau > 0$ is sufficiently small. Therefore, if $x \in \partial S \subset B^{2n}(1)$ and $y \in d\Phi(0)(\tau S)$, and $y = \tau z$ for some $z \in S$, then

$$|h_t((1 + \gamma)\tau x) - d\Phi(0)(\tau z)|$$

$$= |d\Phi(0)((1 + \gamma)\tau x - \tau z) + t(\Phi_k((1 + \gamma)\tau x) - d\Phi(0)((1 + \gamma)\tau x))|$$

$$\geq d\tau(1 + \gamma)x - z - 2\epsilon((1 + \gamma)\tau)(1 + \gamma)\tau$$

$$\geq \tau (dd_1(S, \gamma) - 2\epsilon((1 + \gamma)\tau)(1 + \gamma)),$$

where $dd_1(S, \gamma)$ is a constant depending only on the shape of $S$ and $\gamma$. The last term is greater than 0, provided $\tau$ is so small that

$$dd_1(S, \gamma) > 2\epsilon((1 + \gamma)\tau)(1 + \gamma).$$
Therefore

$$h_t((1 + \gamma) \tau S) \cap d\Phi(0) \tau S = \emptyset. \quad (6.1.8)$$

We want to prove $y \in \Phi_k((1 + \gamma) \tau S)$. For this purpose, we consider Brouwer’s degree

$$\text{deg}(h_t|_{(1 + \gamma) \tau S}; y)$$
as we vary $t$ and $z \in S$. Thanks to (6.1.8), this degree is well defined and continuous over $(t, z) \in [0, 1] \times S$. But, when $t = 0$, $h_0 = d\Phi(0)$ is a linear map and hence $\text{deg}(h_0|_{(1 + \gamma) \tau S}; 0) = 1$. Since $h_1 = \Phi_k$, we have homotopy invariance of the degree

$$\text{deg}(\Phi_k|_{(1 + \gamma) \tau S}; y) = 1$$

for all $y \in S$. This finishes the proof. \hfill $\square$

Then the monotonicity and symplectic invariance of capacities, $\overline{c} \geq c$ and the fact that $d\Phi(0)$ is a linear map imply

$$\overline{c}((1 + \gamma) \tau S) = \overline{c}(\Phi_k((1 + \gamma) \tau S)) \geq c(\Phi_k((1 + \gamma) \tau S))$$

$$\geq c(d\Phi(0)(\tau S)) = \overline{c}(d\Phi(0)(\tau S)).$$

This implies

$$\overline{c}(d\Phi(0)(S)) \leq (1 + \gamma)^2 \overline{c}(S)$$

for all $\gamma > 0$. Hence we have obtained

$$\overline{c}(d\Phi(0)(S)) \leq \overline{c}(S).$$

Combining this with (6.1.6), we have proved

$$\overline{c}(d\Phi(0)(S)) = \overline{c}(S)$$

for all bounded ellipsoids $S$. Then Theorem 6.1.15 implies that $d\Phi(0)$ is either symplectic or anti-symplectic. This finishes the proof of Theorem 6.1.16. \hfill $\square$

### 6.2 Topological Hamiltonian flows and Hamiltonians

A time-dependent Hamilton equation on a symplectic manifold $(M, \omega)$ is the first-order ordinary differential equation

$$\dot{x} = X_H(t, x),$$
where the time-dependent vector field $X_H$ associated with a function $H : \mathbb{R} \times M \to \mathbb{R}$ is given by the defining equation

$$dH_t = X_{H_t} \omega.$$  \hspace{1cm} (6.2.9)

Therefore if we consider functions $H$ that are $C^{1,1}$ so that one can apply the existence and uniqueness theorem of solutions of the above Hamilton equation, the flow $t \mapsto \phi^H_t$, an isotopy of diffeomorphisms, is uniquely determined by the Hamiltonian $H$. Conversely, if a smooth isotopy $\lambda$ of Hamiltonian diffeomorphisms is given, we can obtain the corresponding normalized Hamiltonian $H$ by differentiating the isotopy and then solving (6.2.9). Therefore, in the smooth category this correspondence is bijective.

On the other hand, due to the fact that this correspondence involves differentiating the function and solving Hamilton’s equation, the correspondence gets murkier when the regularity of the Hamiltonian is weaker than $C^{1,1}$ because of the question of the solvability of Hamilton’s equation.

To handle this discrepancy, Müller and Oh in (OhM07) introduced the notion of Hamiltonian limits of smooth Hamiltonian flows and defined the topological Hamiltonian flows as the Hamiltonian limits thereof.

We start with the following definition of the group of symplectic homeomorphisms adopted in (OhM07). We give the compact-open topology on $\text{Homeo}(M)$, which is equivalent to the metric topology induced by the metric

$$d(\phi, \psi) = \max\{d_{C^0}(\phi, \psi), d_{C^0}(\phi^{-1}, \psi^{-1})\}$$

on a compact manifold $M$.

**Definition 6.2.1 (Symplectic homeomorphism group)** Define $\text{Sympeo}(M, \omega)$ to be

$$\text{Sympeo}(M, \omega) := \overline{\text{Symp}(M, \omega)}$$

the $C^0$-closure of $\text{Symp}(M, \omega)$ in $\text{Homeo}(M)$, and call $\text{Sympeo}(M, \omega)$ the symplectic homeomorphism group.

We denote by

$$\mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega), \text{id})$$

the set of smooth Hamiltonian paths $\lambda : [0, 1] \to \text{Symp}(M, \omega)$ with $\lambda(0) = \text{id}$, and equip it with the $C^0$-Hamiltonian topology. This notion was introduced in (OhM07).

**Definition 6.2.2 ($C^0$-Hamiltonian topology)** Let $(M, \omega)$ be a closed symplectic manifold. The $C^0$-Hamiltonian topology of the set $\mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega), \text{id})$
of Hamiltonian paths is the one whose basis is given by the collection of subsets
\[ U(\phi_H, \epsilon_1, \epsilon_2) := \left\{ \phi_{H'} \in \mathcal{P}_{\text{ham}}(\text{Symp}(M, \omega), id) \mid \|\overline{H} \# H'\|_\infty < \epsilon_1, \overline{d}(\phi_H, \phi_{H'}) < \epsilon_2 \right\} \]
of \( \mathcal{P}_{\text{ham}}(\text{Symp}(M, \omega), id) \) for \( \epsilon_1, \epsilon_2 > 0 \) and \( \phi_H \in \mathcal{P}_{\text{ham}}(\text{Symp}(M, \omega), id) \).

**Exercise 6.2.3** Prove that the above Hamiltonian topology is equivalent to the metric topology induced by the metric
\[ d_{\text{ham}}(\lambda, \mu) := \overline{d}(\lambda, \mu) + \text{leng}(\lambda^{-1} \mu), \]
where \( \overline{d} \) is the \( C^0 \) metric on \( \mathcal{P}(\text{Homeo}(M), id) \). (See Proposition 3.10 in (OhM07) for a proof.)

We consider the *developing map*
\[ \text{Dev} : \mathcal{P}_{\text{ham}}(\text{Symp}(M, \omega), id) \rightarrow C^\infty_0([0, 1] \times M, \mathbb{R}). \]
This is defined by the assignment of the normalized generating Hamiltonian \( H \) of \( \lambda \), when \( \lambda = \phi_H : t \mapsto \phi_H^t \). We also consider the maps
\[ \iota_{\text{ham}} : \mathcal{P}_{\text{ham}}(\text{Symp}(M, \omega), id) \rightarrow \mathcal{P}(\text{Symp}(M, \omega), id) \hookrightarrow \mathcal{P}(\text{Homeo}(M, \omega), id). \]
Here the first map is continuous and the second map is the canonical inclusion map.

We call the product map \((\iota_{\text{ham}}, \text{Dev})\) the *unfolding map* and denote the image thereof by
\[ Q := \text{Image}(\iota_{\text{ham}}, \text{Dev}) \subset \mathcal{P}(\text{Homeo}(M), id) \times L^1_{0, \infty}([0, 1] \times M, \mathbb{R}). \quad (6.2.10) \]
Here \( L^1_{0, \infty}([0, 1] \times M, \mathbb{R}) \) is the closure of \( C^\infty_0([0, 1] \times M, \mathbb{R}) \). Then both maps \( \text{Dev} \) and \( \iota_{\text{ham}} \) are Lipshitz with respect to the metric \( d_{\text{ham}} \) on \( \mathcal{P}_{\text{ham}}(\text{Symp}(M, \omega), id) \) by definition and so the unfolding map canonically extends to the closure \( \overline{Q} \) in \( \mathcal{P}(\text{Homeo}(M), id) \times L^1_{0, \infty}([0, 1] \times M, \mathbb{R}) \) in that we have continuous projections
\[ \overline{\pi}_1 : \overline{Q} \rightarrow \mathcal{P}(\text{Homeo}(M), id), \quad (6.2.11) \]
\[ \overline{\pi}_2 : \overline{Q} \rightarrow L^1_{0, \infty}([0, 1] \times M, \mathbb{R}). \quad (6.2.12) \]

We would like to note that by definition we also have the extension of the evaluation map \( \overline{\text{ev}}_1 : \mathcal{P}_{\text{ham}}(\text{Symp}(M, \omega), id) \rightarrow \text{Symp}(M, \omega) \rightarrow \text{Homeo}(M) \) to
\[ \overline{\text{ev}}_1 : \text{Image}(\overline{\pi}_1) \rightarrow \text{Homeo}(M). \quad (6.2.13) \]
The following definition was introduced in (OhM07).

**Definition 6.2.4**  We define the set \( \mathcal{P}^{\text{ham}}(\text{Sympeo}(M, \omega), \text{id}) \) to be
\[
\mathcal{P}^{\text{ham}}(\text{Sympeo}(M, \omega), \text{id}) := \text{Image}(\pi_1) \subset \mathcal{P}(\text{Homeo}(M), \text{id})
\]
and call any element thereof a *topological Hamiltonian path*. We also define
\[
\mathcal{H}_{0}^{(1, \infty)}([0, 1] \times M)
\]
to be the image of \( \tilde{t}_{\text{ham}} \), and call any element therein a *topological Hamiltonian*.

It is natural to ask whether the projections \( \pi_1 \) and \( \pi_2 \) are one-to-one. This question turns out to touch the heart of \( C^0 \)-symplectic topology and involves some function theory on symplectic manifolds. In Section 22.6, we will study this injectivity question in detail.

Unraveling the above definition, for any given element \( H \in \mathcal{H}_{0}^{(1, \infty)}([0, 1] \times M) \), there exists a sequence of smooth Hamiltonians \( H_i \) such that
\[
(1) \quad H_i \to H \text{ in } L^{(1, \infty)} \text{ topology, and}
(2) \quad \text{Both } \phi_{H_i} \text{ and } \phi_{H_i}^{-1} \text{ uniformly converge on } [0, 1] \times M.
\]
We call any such sequence an *approximating sequence* of the topological Hamiltonian \( H \).

We denote \( C^0(M) \) the set of continuous functions on \( M \) with the \( C^0 \)-norm \( \| \cdot \|_{C^0} \).

**Definition 6.2.5**  We define \( L^1([0, 1], C^0(M)) \), which we call the space of \( L^1 \)-functions on \( [0, 1] \) with values lying in \( C^0(M) \), to be the set of maps \( H : t \in [0, 1] \to C^0(M) \) defined almost everywhere such that the function
\[
\text{tmint}(H)(t, x) := \int_0^t H(s, x)ds
\]
is jointly continuous in \( (t, x) \).

We have the canonical embedding
\[
L^1([0, 1], C^0(M)) \hookrightarrow L^{(1, \infty)}([0, 1] \times M).
\]

In this regard, we first prove the following theorem.

**Theorem 6.2.6**  Let \( H \in \mathcal{H}_{0}^{(1, \infty)}([0, 1] \times M) \subset L^{(1, \infty)}([0, 1] \times M) \). Then the following statements hold.
\[
(1) \quad H \text{ lies in } L^1([0, 1], C^0(M)), \text{ i.e., } \mathcal{H}_{0}^{(1, \infty)}([0, 1] \times M) \subset L^1([0, 1], C^0(M)).
\]
(2) The function \( \text{osc}(H)(t) := \text{osc}(H_t(\cdot)) \) is defined a.e. in \( t \in [0, 1] \) and lies in \( L^1([0, 1], \mathbb{R}) \).

**Proof** Let \( H_i \) be an approximating sequence of \( H \). Then we have the convergence

\[
\int_0^1 \text{osc}(H_i - H) dt \to 0
\]

by definition of the approximating sequence above. In particular,

\[
\int_0^1 \text{osc}(H_i - H_j) dt \to 0 \quad \text{as } i, j \to \infty.
\]

We first examine the function \( (t, x) \mapsto \int_0^t H(s, x) ds \). Consider the functions \( \text{osc}(H_i) : [0, 1] \to \mathbb{R}_+ \) defined by

\[
\text{osc}(H_i)(t) := \text{osc}(H_i, t) \quad \text{for } t \in [0, 1],
\]

where we recall that \( \text{osc}(h) = \max h - \min h \) for a function \( h : M \to \mathbb{R} \). Since \( H_i \to H \) in \( L^{1,\infty} \), the sequence \( \text{osc}(H_i) \) is a Cauchy sequence in \( L^1([0, 1], \mathbb{R}) \).

Note that

\[
|\text{osc}(H_i) - \text{osc}(H_j)| \leq \text{osc}(H_i - H_j).
\]  

(6.2.14)

We also consider \( \text{osc}(H_i - H_j) \). By the \( L^{1,\infty} \) convergence of \( H_i \), we have

\[
\text{osc}(H_i - H_j) \to 0 \quad \text{in } L^1([0, 1], \mathbb{R}) \quad \text{as } i, j \to \infty.
\]

In particular, (6.2.14) implies that there exists a subsequence of \( H_{i_k} \) such that \( \text{osc}(H_{i_k}) \) converges a.e. in \( [0, 1] \) as \( k \to \infty \), and

\[
\text{osc}(H_{i_k} - H_{i_j})(t) \to 0 \quad \text{a.e. in } [0, 1] \quad \text{as } k, j \to \infty.
\]

We denote

\[
A = \{ t \in [0, 1] \mid \text{osc } H_{i_k}(t) \text{ converges} \},
\]

which has a full measure. Let \( t \in A \). Then, since we have

\[
\text{osc}(H_{i_k} - H_{i_j})(t) = \max_{x \in M} (H_{i_k,t} - H_{i_j,t}) - \min_{x \in M} (H_{i_k,t} - H_{i_j,t})
\]  

(6.2.15)

by the definition of \( \text{osc} \), \( H_{i_k}(\cdot, t) : M \to \mathbb{R} \) is a Cauchy sequence in \( C^0 \) topology, and hence converges uniformly to a continuous function \( H_{\infty,t} \) at each \( t \in A \). If we set \( H_{\infty}(t, x) = H_{\infty,t}(x) \) for \( (t, x) \in A \times M \) whenever \( H_{\infty,t} \) is defined, \( H_{\infty} \) is a function almost everywhere defined. From (6.2.15) combined with Fatou’s lemma, we derive

\[
\int_0^1 \text{osc}(H_{i_k} - H_{\infty})(t) dt \leq \liminf_{j \to \infty} \int_0^1 \text{osc}(H_{i_k} - H_{i_j})(t) dt.
\]
Then, by letting $k \to \infty$, we derive
\[
\lim_{k \to \infty} \int_0^1 \text{osc}(H_{i_k} - H_\infty)(t)dt = 0.
\]
This implies that $H_\infty$ is also a $L^{(1,\infty)}$-limit of $H_{i_k}$. By the uniqueness of the $L^{(1,\infty)}$-limits, it follows that $H = H_\infty$ is an $L^{(1,\infty)}$-function. It then follows that $H(t, \cdot)$ is defined and continuous, and $H_i(t, \cdot) \to H(t, \cdot)$ uniformly for almost all $t \in [0, 1]$. This proves $\mathcal{H}_0^{(1,\infty)}([0, 1] \times M) \subset L^1([0, 1], C^0(M))$.

In the course of the above proof, we also prove that $\text{osc}(H(t)) = \text{osc}(H(\cdot, t))$ is defined a.e. and lies in $L^1([0, 1], \mathbb{R})$.

\[\square\]

### 6.3 Uniqueness of the topological Hamiltonian and its flow

In this section, we study the fundamental uniqueness theorems on the topological Hamiltonian and on its flow.

The uniqueness of the flow can be rephrased as the injectivity of the developing map
\[\pi_2 : \overline{Q} \to L^{(1,\infty)}_0([0, 1] \times M, \mathbb{R}).\]

**Theorem 6.3.1** Let $\lambda_i = \phi_{H_i}, \lambda'_i = \phi_{H'_i} \in \mathcal{P}^{\text{Ham}}(\text{Symp}(M, \omega), \text{id})$ be two sequences of smooth Hamiltonian paths such that

1. both $H_i$ and $H'_i$ converge in $L^{(1,\infty)}$ topology and $\|H_i\#H'_i\| \to 0$; and
2. $\phi_{H_i}, \phi_{H'_i}$ uniformly converge on $[0, 1] \times M$.

Then $\lim_i \phi_{H_i} = \lim_i \phi_{H'_i}$.

**Proof** We first note that, since both $\phi_{H_i}$ and $\phi_{H'_i}$ uniformly converge, the conclusion is equivalent to the uniform convergence
\[\lim_i (\phi'_{H_i})^{-1} \phi'_{H'_i} \to \text{id}\]
uniformly over $t \in [0, 1]$ as $i \to \infty$. Therefore, by considering $\overline{H_i}\#H'_i$, the proof is reduced to the following special case. Suppose $H_i$ is a sequence satisfying

1. $\|H_i\| \to 0$,
2. $\phi_{H_i}$ uniformly converges on $[0, 1] \times M$.

Then $\phi_{H_i}$ converges to the constant identity path in $C^0$ topology.

In the remainder of the proof, we prove this statement. Denote $\lambda = \lim_{i \to \infty} \phi_{H_i}$ lying in $\mathcal{P}(\text{Homeo}(M), \text{id})$ and consider its values $\lambda(t)$. We would like to show that $\lambda(t) \equiv \text{id}$ for all $t$. Suppose the contrary and consider
We would like to show that $t_0 = 1$. Suppose to the contrary that $t_0 < 1$. Since $\lambda(t_1) \neq id$ for some $t_1 > 0$ and $\lambda(0) = id$, we know that $t_0 > 0$. Then we can choose $\delta > 0$ sufficiently small that $0 < t_0 + \delta < 1$ and $\psi = \lambda(t_0 + \delta) \neq id$. We note that $\psi$ must be continuous since it is a uniform limit of continuous maps $\phi_{H_i^{t_0+\delta}}$. Then we can find a small closed ball $B$ such that

$$B \cap \psi(B) = \emptyset.$$ 

Since $B$ and hence $\psi(B)$ is compact and $\phi_{H_i^{t_0+\delta}} \to \psi$ uniformly, we have

$$B \cap (\phi_{H_i^{t_0+\delta}})(B) = \emptyset$$

for all sufficiently large $i$. By definition of the Hofer displacement energy $e$, we have $e(B) \leq \|\phi_{H_i^{t_0+\delta}}\|$. By the energy–capacity inequality from Theorem 5.4.4 we know that $e(B) > 0$ and hence

$$0 < e(B) \leq \|H_i^{t_0+\delta}\|$$

for all sufficiently large $i$. Here $H_i^u$ is the Hamiltonian generating the reparameterized subpath of $\phi_{H_i}$

$$t \in [0, 1] \mapsto \phi_{H_i}^u,$$

which we already know from the explicit formula $H_i^u(t, x) = u H_i(ut, x)$.

Therefore we have derived

$$0 < e(B) \leq \int_0^1 \text{osc}(H_i^u) dt = \int_0^1 u(\max H_{i, ut} - \min H_{i, ut}) dt$$

$$= \int_0^{u(\max H_{i, t} - \min H_{i, t})} dt \leq \|H_i\| \to 0$$

for $u = t_0 + \delta$, which gives rise to a contradiction. Hence $t_0 = 1$, which finishes the proof. \hfill \Box

Next we turn to the injectivity of the map $\bar{\pi}_1 : \bar{Q} \to \mathcal{P}(\text{Homeo}(M), id)$ in (6.2.11), which we phrase as the ‘uniqueness of the topological Hamiltonian’.

**Question 6.3.2** Assume that $H_i, H'_i : [0, 1] \times M \to \mathbb{R}$ are two sequences of Hamiltonians that satisfy the normalization condition. Suppose that the Hamiltonian paths $\phi_{H_i}, \phi_{H'_i}$ converge uniformly and satisfy

$$d(\phi_{H_i}, \phi_{H'_i}) \to 0.$$ 

Then is it true that $\lim_{i \to \infty} H_i = \lim_{i \to \infty} H'_i$ as an $L^{(1, \infty)}$ function?
This question was posed in (OhM07), and has been answered affirmatively by the following theorem.

**Theorem 6.3.3** (Viterbo, Buhovsky–Seyfaddini) Assume that \( H_i : [0, 1] \times M \to \mathbb{R} \) is a sequence of Hamiltonians. Then \( H_i \to h(t) \) in \( L^{(1,\infty)} \) topology, where \( h : [0, 1] \to [0, 1] \) is an \( L^1 \)-function, provided that

1. \( H_i \to h(t) \) converges in \( L^{(1,\infty)} \) topology, where \( h : [0, 1] \to [0, 1] \) is an \( L^1 \)-function; and
2. \( \phi_{H_i} \) converges uniformly to the constant path id.

In particular, if the \( H_i \) are mean-normalized, then \( h \equiv 0 \).

This theorem was first proved in the \( C^0 \) context (or in the \( L^\infty \) context) by Viterbo (Vi06) for the closed case and then extended to the compactly supported case on open manifolds in (Oh07). This original proof relies on some \( C^0 \)-Lagrangian intersection theory, which itself is very interesting.

We present an elegant proof given by Buhovsky and Seyfaddini (BS10) that is based on an ingenious application of the classical Lebesgue differentiation theorem. The standard Lebesgue differentiation theorem in the literature is stated for the real-valued functions. One crucial ingredient in their proof is a differentiation theorem for a function lying in \( L^1([0, 1], C^0(M)) \) defined in the sense of Definition 6.2.5. We state a version of the Lebesgue differentiation theorem in this generality.

First we recall a few basic definitions in the classical analysis.

**Definition 6.3.4** Let \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \). The Lebesgue set \( L_f \) of \( f \) is defined to be

\[
L_f = \left\{ x \in \mathbb{R}^n \mid \lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| dy = 0 \right\},
\]

where \( B_r(x) \) is a ball of radius \( r \) centered at \( x \) and \( m(B_r(x)) \) is the Lebesgue measure of \( B_r(x) \).

The following is the classical differentiation theorem in function theory on \( \mathbb{R}^n \) whose proof can be found in any standard real analysis book, e.g., in (Fol99).

**Theorem 6.3.5** (Lebesgue differentiation theorem) Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a real-valued function defined almost everywhere. Denote the complement of \( L_f \) in \( \mathbb{R}^n \) by \( (L_f)^c \). If \( f \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}) \), then \( m((L_f)^c) = 0 \).
We will need a more general form of differentiation theorem. In this regard, we recall the following notion from (Fol99) and state a more general form of the Lebesgue differentiation theorem.

**Definition 6.3.6** A family of sets \( \{E_r\}_{r>0} \) of Borel subsets of \( \mathbb{R}^n \) is said to **shrink nicely** to \( x \in \mathbb{R}^n \) if

1. \( E_r \subset B_r(x) \) for each \( r \) and
2. there is a constant \( \alpha > 0 \), independent of \( r \), such that the inequality \( m(E_r) > \alpha m(B_r(x)) \) holds.

Here the sets \( E_r \) need not contain \( x \) itself.

**Theorem 6.3.7** Suppose \( f \in L^1_{\text{loc}} \). For every \( x \) in the Lebesgue set of \( f \), in particular for almost every \( x \), we have

\[
\lim_{t \to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0, \quad (6.3.16)
\]

\[
\lim_{t \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x) \quad (6.3.17)
\]

for every family \( \{E_r\}_{r>0} \) that shrinks nicely to \( x \).

We will apply this theorem to the \( L^1 \)-function \( t \in [0, 1] \subset \mathbb{R} \to H(t, x) \in \mathbb{R} \) for each \( x \in M \) and \( H \in \mathcal{H}_0^{(1,\infty)}([0, 1] \times M, \mathbb{R}) \) and the family of sets

\[
E_h(t) := [t, t+h], \quad h > 0, \ t \in [0, 1].
\]

We remark that the family \( \{E_h(t)\}_{h>0} \) shrinks nicely to \( t \). We state the following version of the differentiation theorem used in (BS10).

**Proposition 6.3.8** (Buhovsky–Seyfaddini) Consider \( H \in \mathcal{H}_0^{(1,\infty)}([0, 1] \times M) \subset L^{(1,\infty)}([0, 1] \times M) \). Then we have

\[
\lim_{h \to 0^+} \frac{1}{h} \int_{t}^{t+h} \osc(H_s - H_t) ds = 0
\]

almost everywhere in \( t \in [0, 1] \).

**Proof** Let \( H_i \) be an approximating sequence of \( H \). Then we have the convergence

\[
\int_0^1 \osc(H_i - H) dt \to 0
\]
and, in particular,
\[ \int_0^1 \text{osc}(H_i - H_j) dt \to 0 \quad \text{as} \; i, \; j \to \infty. \]

We also have
\[ \int_t^{t+h} \text{osc}(H_s - H_t) ds \leq \int_t^{t+h} \text{osc}(H_{i,s} - H_{i,t}) ds + \int_t^{t+h} \text{osc}(H_s - H_{i,s}) ds \]
\[ + \int_t^{t+h} \text{osc}(H_{i,t} - H_t) ds. \tag{6.3.18} \]

Since \( H_i \) is smooth,
\[ \lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \text{osc}(H_{i,s} - H_{i,t}) ds = 0 \]
for all \( t \in [0, 1] \). On the other hand, as we have shown in Theorem 6.2.6 (2), the function \( \text{osc}(H) \) lies in \( L^1([0, 1], \mathbb{R}) \) and \( \text{osc}(H - H_i) \to 0 \) in \( L^1([0, 1], \mathbb{R}) \). Denote \( F_i = H - H_i \) and apply Theorem 6.3.5 to \( f_i := \text{osc}(F_i) \) for each \( i \) to prove
\[ \lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} |\text{osc}(F_{i,s}) - \text{osc}(F_{i,t})| ds = 0 \]
almost everywhere in \( t \). Denote by \( L_{f_i} \) the Lebesgue set of \( f_i \) and consider the set
\[ \bigcap_{i=1}^{\infty} L_{f_i} =: A \subset [0, 1]. \]

This set has full measure in \([0, 1]\). Furthermore since \( f_i \to 0 \) in \( L^1 \), it converges to 0 pointwise almost everywhere in \([0, 1]\), say on \( B \subset [0, 1] \) of full measure.

We claim that
\[ \lim_{h \to 0^+} \int_t^{t+h} \text{osc}(H_s - H_t) ds = 0 \tag{6.3.19} \]
at every \( t \in A \cap B \). At every \( t \in A \), we have
\[ \lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \text{osc}(F_{i,s}) ds \leq \lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} |\text{osc}(F_{i,s}) - \text{osc}(F_{i,t})| ds \]
\[ + \lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \text{osc}(F_{i,t}) ds \]
\[ = \lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} |f_i(s) - f_i(t)| ds + f_i(t) = f_i(t) \]
at such \( t \). Substituting this into (6.3.18), we obtain
\[
\lim_{h \to 0^+} \int_t^{t+h} \text{osc}(H_s - H_t)ds \leq f_i(t)
\]
for all \( i \). Since \( t \in B \), \( f_i(t) \to 0 \). By letting \( i \to \infty \), we obtain
\[
\lim_{h \to 0^+} \int_t^{t+h} \text{osc}(H_s - H_t)ds = 0.
\]
Since \( A \cap B \) has full measure, this finishes the proof. \( \square \)

Now we are ready to prove Theorem 6.3.3. One crucial idea of (BS10) lies in their study of the set of null Hamiltonians
\[
\mathcal{H}_{\text{null}} = \{ H \in \mathcal{H}_0^{1, \infty}([0, 1] \times M, \mathbb{R}) \mid \phi_H \equiv \text{id} \} = (\bar{\iota}_{\text{ham}})^{-1}(\text{id})
\]
and its autonomous analog
\[
\mathcal{H}_{\text{null}}^{\text{aut}} = \{ H \in \mathcal{H}_{\text{null}} \mid H \text{ is time-independent} \} \subset C^0(M).
\]
The uniqueness can be rephrased as \( \mathcal{H}_{\text{null}} = \{ 0 \} \). In the rest of this section, we will prove this statement.

The proof consists of a series of lemmata and exercises.

**Lemma 6.3.9** If \( H \in \mathcal{H}_{\text{null}} \cap C^\infty([0, 1] \times M) \), then \( H \equiv 0 \).

**Proof** Let \( H_i \) be a sequence of smooth Hamiltonians that represents \( H \), i.e., \( \|H_i - H\| \to 0 \) and \( \phi_{H_i} \to \text{id} \). By Theorem 6.3.1, we have \( \phi_H \equiv \text{id} \). By differentiating the flow \( \phi_H \), we obtain \( X_H \equiv 0 \), which then implies that \( dH_t = 0 \) and hence \( H_t \equiv \text{constant for all } t \). By the normalization condition, this implies \( H_t \equiv 0 \) for all \( t \). \( \square \)

**Exercise 6.3.10** Prove the following properties of \( \mathcal{H}_{\text{null}} \).

1. \( \mathcal{H}_{\text{null}} \) is a closed subset of \( L^{1, \infty} \), but not necessarily a vector subspace.
2. \( \mathcal{H}_{\text{null}} \) is closed under the operation \( H \mapsto H^\rho \), where \( H^\rho(t, x) = \rho^\prime(t)H(\rho(t), x) \) for any smooth increasing function \( \rho : [0, 1] \to \mathbb{R} \).
3. \( \mathcal{H}_{\text{null}} \) is closed under the operations of taking the sum and taking the minus.

**Lemma 6.3.11** \( \mathcal{H}_{\text{null}}^{\text{aut}} \) is a vector space over \( \mathbb{R} \) and the action of \( \text{Symp}(M, \omega) \) on \( \mathcal{H}_{\text{null}}^{\text{aut}} \) \( h \mapsto \psi^*h = h \circ \psi \) preserves \( \mathcal{H}_{\text{null}}^{\text{aut}} \).

**Proof** For the second statement, let \( H_i \) be a sequence representing \( h \in \mathcal{H}_{\text{null}}^{\text{aut}} \). Then it follows that the Hamiltonian \( H_i(t, \psi(x)) \) represents \( h \circ \psi \), since
\( H_i(t, \psi(x)) \) generates the flow \( \psi \circ \phi'_H \circ \psi^{-1} \). Since \( \phi_H \to \text{id} \), it follows that \( \psi \circ \phi'_H \circ \psi^{-1} \to \text{id} \). Therefore \( h \circ \psi \in \mathcal{H}^{\text{aut}}_{\text{null}} \).

We now prove the first statement. Obviously \( 0 \in \mathcal{H}^{\text{aut}}_{\text{null}} \). Let \( h, k \in \mathcal{H}^{\text{aut}}_{\text{null}} \), and let \( H_i, K_i \), respectively, be their associated approximating sequences. Then the Hamiltonian \( H \# K_i \) represents the flow \( \phi_H \circ \phi_{K_i} \). Since \( \phi_H, \phi_{K_i} \to \text{id} \), \( \phi_H \circ \phi_{K_i} \to \text{id} \). On the other hand, we have by definition

\[
(H_i \# K_i)(t, x) = H_i(t, x) + K_i(t, (\phi_H^{-1}(x)).
\]

Since \( H_i \to h, K_i \to k \) in \( L^{(1, \infty)} \), it follows that \( H_i \# K_i \to h + k \).

**Exercise 6.3.12** Prove the function \( (t, x) \mapsto K_i(t, (\phi_H^{-1}(x)) \) converges to \( k \) in \( L^{(1, \infty)} \).

Similarly, we prove \( -h \in \mathcal{H}^{\text{aut}}_{\text{null}} \) by recalling that \( \overline{H}_i(t, x) = -H_i(t, \phi_H(x)) \) represents the inverse flow \( \phi_H^{-1} \). Since \( \phi_H \to \text{id} \), we have \( \phi_H^{-1} \to \text{id} \), and thus \( H_i \to h \) and \( -H_i(t, \phi_H(x)) \to -h \) in \( L^{(1, \infty)} \). This finishes the proof.

**Lemma 6.3.13** Let \( H \in \mathcal{H}_{\text{null}} \). Then, for almost every \( t \in [0, 1] \) at which \( H_t \in C^0(M) \), the time-independent function \( h(x) := H_t(x) = H(t, x) \) lies in \( \mathcal{H}^{\text{aut}}_{\text{null}} \).

**Proof** By Proposition 6.3.8, for almost every \( t \in [0, 1] \), we have

\[
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} ||H_s - H_t||_{\infty} \, ds = 0.
\]

For sufficiently large \( N \in \mathbb{N} \), we consider the function \( G_N(s, x) = (1/N)H(t + s/N, x) \), which corresponds to \( H^\rho \) with \( \rho(s) = t + s/N \). Therefore \( G_N \in \mathcal{H}_{\text{null}} \). Since \( \mathcal{H}_{\text{null}} \) is closed under the sum operation by Exercise 6.3.10 (3), we also have \( H_N := NG_N \in \mathcal{H}_{\text{null}} \). But \( NG_N(s, x) = H(t + s/N, x) \). Setting \( \varepsilon = 1/N \), we obtain

\[
\int_0^{1} ||H_N(s, \cdot) - h(\cdot)||_{\infty} \, ds = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} ||H(\tau, \cdot) - H(t, \cdot)||_{\infty} \, d\tau \to 0,
\]

where \( h \) is the function defined by \( h(x) = H_t(x) \) for the given fixed \( t \in [0, 1] \). Therefore \( H_N \to h \) in \( L^{(1, \infty)} \) as \( N \to \infty \). Since \( \mathcal{H}_{\text{null}} \) is a closed subset of \( L^{(1, \infty)} \), \( h \in \mathcal{H}_{\text{null}} \), which in turn implies that \( h \in \mathcal{H}^{\text{aut}}_{\text{null}} \) since it is time-independent.

The following is another key lemma of Buhovsky and Seyfaddini that employs the tools from classical analysis in a novel way.
Lemma 6.3.14  If $h \in \mathcal{H}_{\text{null}}^\ast$ then $h \equiv 0$.

Proof  We prove this by contradiction. Suppose $h \neq 0$. Under this hypothesis we will construct a smooth non-zero function lying in $\mathcal{H}_{\text{null}}^\ast \subset \mathcal{H}_{\text{null}}$ that will contradict Lemma 6.3.9.

Since $h$ is not constant and $M$ is connected, $h$ is not locally constant. Therefore there exists $x_0 \in M$ such that $h$ is not constant on any open neighborhood thereof. Choose a Darboux neighborhood $U$ at $x_0$. For each $y_0 \in U$ with $h(y_0) \neq h(x_0)$, we can choose $\phi \in \text{Ham}_U(M, \omega)$ with $\phi(x_0) = y_0$. We consider the function $k = h \circ \phi - h$. By the choice of $y_0$ and $\phi$, $k$ is a non-constant function and is contained in $\mathcal{H}_{\text{null}}^\ast$ by Lemma 6.3.11. Moreover, $k = 0$ near $\partial U$. Since $k = 0$ near $\partial U$, we can extend $k$ to the whole of $M$ by 0 outside $U$. We define $\mathcal{L}$ as the $C^0$ closure of the linear span of all functions on $M$ of the form $\psi^*k = k \circ \psi$ with $\text{supp}k \subset U$ and $\psi \in \text{Ham}_U(M, \omega)$. Then $\mathcal{L} \subset \mathcal{H}_{\text{null}}^\ast$. We now claim that $\mathcal{L}$ contains a non-zero smooth element.

Since $k = 0$ outside $U$ and $U$ is a Darboux chart, we may assume that $U \subset \mathbb{R}^{2n}$ and need only prove the statement on an open $U \subset \mathbb{R}^{2n}$. We keep the same notation for $\mathcal{L}$, which is the $C^0$ closure of the linear span of the functions of the form $\psi^*k$ with $\psi \in \text{Ham}_U(\mathbb{R}^{2n})$.

We denote $k_v(x) = k(x - v)$ for $v \in \mathbb{R}^{2n}$. We next prove that, if $||v||$ is sufficiently small, then $k_v \in \mathcal{L}$. Take a neighborhood $W$ of supp $k$ with $\overline{W} \subset U$ and pick $v$ with $||v||$ so small that $\overline{W} + tv \subset U$ for all $0 \leq t \leq 1$. We pick a cut-off function $\rho$ such that supp $\rho \subset U$ and $\rho \equiv 1$ on $\bigcup_{t \in [0,1]} \overline{W} + tv$. Then we consider the function $g_v(x) = \omega(v, x)\rho$. The Hamiltonian flow $\phi_{g_v}^t$, on $\bigcup_{t \in [0,1]} \overline{W} + tv$ is nothing but the flow $t \mapsto x + tv$ for all $x \in W$ and hence on supp $k$. Therefore we have $((\phi_{g_v}^1)^{-1})^*k = k \circ (\phi_{g_v}^1)^{-1} = k_v$. This proves that $k_v \in \mathcal{L}$.

Now we choose a smooth function $\chi$ whose support is contained in a sufficiently small neighborhood of the origin in $\mathbb{R}^{2n}$ and consider the convolution $k \ast \chi$ defined by

$$k \ast \chi(x) = \int_{\mathbb{R}^{2n}} k(x - v)\chi(v)dv.$$ 

Then we claim that $k \ast \chi \in \mathcal{L}$: by taking the approximations of the integral by Riemann sums, we can write $k \ast \chi$ as the $C^0$-limit of the sum

$$\sum_{i=1}^{m} c_i k_{v_i},$$ 

Exercise 6.3.15  Prove this $C^0$-convergence.

Then, since $\sum_{i=1}^{m} c_i k_{v_i} \in \mathcal{L}$, we have $k \ast \chi \in \mathcal{L}$ by the definition of $\mathcal{L}$.

Finally we take the mollifier smoothing $\chi_\epsilon$ of the Dirac $\delta_0$ function so that $k = \lim_{\epsilon \to 0} k \ast \chi_\epsilon$ in $C^0$. Since $k \neq 0$, if $\epsilon$ is sufficiently small $k \ast \chi_\epsilon$ is a non-zero
smooth function. By the above claim, each $k \ast \chi_\varepsilon$ lies in $L \subset \mathcal{H}^{\text{aut}}_{\text{null}}$. This finishes the proof of Lemma 6.3.14.

Wrap-up of the proof of Theorem 6.3.3

Let $H \in \mathcal{H}_{\text{null}}$. Then at each Lebesgue point $t$ of $H \in L^1([0, 1], C^0(M))$ we have $H_t \in \mathcal{H}^{\text{aut}}_{\text{null}}$ and so $H_t \equiv 0$. Since the set of Lebesgue points has full measure in $[0, 1]$, $H \equiv 0$ in $L^1([0, 1], C^0(M))$. Therefore $H = 0$ in $L^{(1, \infty)}([0, 1] \times M, \mathbb{R})$. This finishes the proof.

A small modification of the above proof will also give rise to a locality of the topological Hamiltonians. We refer the reader to (BS10) for the details. This locality result was originally proved in (Oh07) in the $C^0$ context by a different method using the Lagrangian intersection theorem of the conormal to open subsets following the spirit of Viterbo’s proof.

**Theorem 6.3.16** (Oh (Oh07), Buhovsky–Seyfaddini (BS10)) Suppose that $H \in \mathcal{H}^{(1, \infty)}$ and its topological Hamiltonian flow satisfies $\phi^t_H|_U \equiv \text{id}|_U$. Then $H(t, x) = c(t)$ for all $x \in [0, 1] \times U$ for some constant $c(t)$ at each $t$.

### 6.4 The homeomorphism group

The authors of (OhM07) introduced the notion of **Hamiltonian homeomorphisms** given by

$$\text{Hameo}(M, \omega) = \{h \in \text{Homeo}(M) \mid h = e_1(\lambda), \lambda \in \mathcal{P}^{\text{ham}}(\text{Sympeo}(M, \omega), \text{id})\}$$

in the context of $L^{(1, \infty)}$ topology of Hamiltonian paths. The following normality was proved in the $L^{(1, \infty)}$ context in (OhM07).

**Theorem 6.4.1** $\text{Hameo}(M, \omega)$ is a path-connected normal subgroup of $\text{Sympeo}(M, \omega)$ (with respect to the subspace topology of $\text{Homeo}(M)$).

The proof of this theorem given in (OhM07) is rather delicate largely due to some issues related to the interplay between the $L^{(1, \infty)}$ limit of Hamiltonians and the $C^0$ limit of Hamiltonian paths. On the other hand, one can repeat the construction given in the previous section and this in the $C^0$ context on $[0, 1] \times M$ in a simpler way than in the $L^{(1, \infty)}$ context. In this way, we have two a-priori different normal subgroups of $\text{Sympeo}(M, \omega)$, which we temporarily denote by $\text{Hameo}_{(1, \infty)}(M, \omega)$ and $\text{Hameo}_\infty(M, \omega)$. 
But the following theorem was proved by Müller (Mue08)

**Theorem 6.4.2** (Müller) We have  

\[
\text{Hameo}_{(1,\infty)}(M, \omega) = \text{Hameo}_{\infty}(M, \omega).
\]

On this basis, we can just write either Hamiltonian homeomorphism group as \( \text{Hameo}(M, \omega) \) from now on. Thanks to Theorem 6.4.2 above, we can use the \( C^0 \) as limits of smooth Hamiltonians in the proof of Theorem 6.4.1. Then the proofs of normality and path-connectedness become very simple.

**Proof of Theorem 6.4.1** We will use the description of \( \text{Hameo}(M, \omega) \) in terms of \( \text{Hameo}_{\infty}(M, \omega) \) in this proof, which is based on Theorem 6.4.2. Let \( h \in \text{Hameo}_{\infty}(M, \omega) \) and \( g \in \text{Sympeo}(M, \omega) \). By the definitions of \( \text{Hameo}_{\infty}(M, \omega) \) and \( \text{Sympeo}(M, \omega) \), there exist a sequence \( H_i \) of smooth Hamiltonians and a sequence \( \psi_i \in \text{Symp}(M, \omega) \) such that \( (\phi_{H_i}, H_i) \) and \( \psi_i \) converge in \( C^0 \) topology, and

\[
\lim_{i \to \infty} \phi_{H_i}^1 = h, \quad \lim_{i \to \infty} \psi_i = g.
\]

Obviously \( H_i \circ \psi_i \to H \circ g \) and \( \psi_i^{-1} \phi_{H_i} \psi_i = \phi_{H_i \circ \psi_i} \) converge in \( C^0 \) topology, which implies that first \( H \circ g \) is a topological Hamiltonian and then \( H_i \circ \psi_i \) is an approximating sequence thereof. Furthermore,

\[
\psi_i^{-1} \phi_{H_i} \psi_i = \text{ev}_1(\psi_i^{-1} \phi_{H_i} \psi_i)
\]

and

\[
\lim_{i \to \infty} \psi_i^{-1} \phi_{H_i} \psi_i = g^{-1} h g.
\]

Hence, by definition, we have proved \( g^{-1} h g \in \text{Hameo}_{\infty}(M, \omega) \), which proves the normality of \( \text{Hameo}_{\infty}(M, \omega) \) in \( \text{Sympeo}(M, \omega) \). The path-connectedness is immediately evident since the above proof shows that any element \( h \in \text{Hameo}_{\infty}(M, \omega) \) can be connected to the identity via a path lying in \( \text{Hameo}_{\infty}(M, \omega) \) that is connected in \( \text{Sympeo}(M, \omega) \). \( \square \)

Similarly to the case of the interval \([0, 1]\), we can define a topological Hamiltonian path on \([a, b]\) with \( b > a \)

\[
\lambda : [a, b] \to \text{Hameo}(M, \omega)
\]

to be a path such that

\[
\lambda \circ (\lambda(a))^{-1} \in \mathcal{P}^{\text{ham}}_{\{a, b\}}(\text{Sympeo}(M, \omega), \text{id}),
\]

where \( \mathcal{P}^{\text{ham}}_{\{a, b\}}(\text{Sympeo}(M, \omega), \text{id}) \) is defined in the same way as \( \mathcal{P}^{\text{ham}}(\text{Sympeo}(M, \omega), \text{id}) \) with \([0, 1]\) replaced by \([a, b]\).

We believe that the following conjecture is true.
Conjecture 6.4.3  The subset $C^0_{ham}(M, \omega)$ is a proper subset of $C^0(M)$.

Using the uniqueness result (BS10) for topological Hamiltonians, one can extend the definition of the Hofer length function to the topological Hamiltonian paths, and define its associated intrinsic distance function on $\text{Hameo}(M, \omega)$ by extending the one on $\text{Ham}(M, \omega)$. We refer the reader to (Oh10) for further details and other related discussions.

The following theorem is the topological Hamiltonian analog to the well-known theorem on the Hofer norm on $\text{Ham}(M, \omega)$ (H90).

Theorem 6.4.4  Let $g, h \in \text{Hameo}(M, \omega)$. Then the extended Hofer norm function

$$|| \cdot || : \text{Hameo}(M, \omega) \to \mathbb{R}_+$$

is continuous in the Hamiltonian topology, and satisfies the following properties:

1. (Symmetry)  $||g|| = ||g^{-1}||$
2. (Bi-invariance)  $||gh|| = ||hg||$
3. (Triangle inequality)  $||gh|| \leq ||g|| + ||h||$
4. (Symplectic invariance)  $||\psi^{-1}g\psi|| = ||g||$ for any $\psi \in \text{Sympeo}(M, \omega)$
5. (Nondegeneracy)  $g = \text{id}$ if and only if $||g|| = 0$.

However, Stefan Müller (Mue08) posed the following question.

Question 6.4.5 (S. Müller)  Denote by $|| \cdot ||_{\text{Ham}}$ and $|| \cdot ||_{\text{Hameo}}$ the Hofer norm on $\text{Ham}(M, \omega)$ and the extended Hofer norm on $\text{Hameo}(M, \omega)$, respectively. Let $\phi \in \text{Ham}(M, \omega) \subset \text{Hameo}(M, \omega)$. Does the identity

$$||\phi||_{\text{Ham}} = ||\phi||_{\text{Hameo}}$$

hold in general?

Some light can be shed on this question by relating it to the general construction of path metric spaces $(X, d_\ell)$ starting from a general metric space $(X, d)$ from the point of view of Chapter 1 (Gr88).

The following conjecture is one of the important motivations to understand the Hamiltonian dynamics in the $C^0$ level.

Conjecture 6.4.6  The Calabi homomorphism $\text{Cal} : \text{Ham}(D^2, \partial D^2) \to \mathbb{R}$ is extended to a homomorphism
that is continuous in Hamiltonian topology.

In (Oh10), it was shown that validity of this conjecture together with the smoothing theorem (Oh06b), (Sik07) would imply properness of $\text{Hameo}(D^2, \partial D^2)$ in $\text{Homeo}^\Omega(D^2, \partial D^2)$ and hence leads to a proof of the non-simplicity of $\text{Homeo}^\Omega(D^2, \partial D^2)$. In Part 4, after we have explained the construction of spectral invariants and the notion of Entov–Polterovich quasimorphisms (EnP03), we will discuss this conjecture further.
PART 2

Rudiments of pseudoholomorphic curves
In this chapter, we perform geometric calculations involving general pseudoholomorphic curves in the context of almost-Kähler manifolds \((M, \omega, J)\). We first derive some general facts on the harmonic energy density of pseudoholomorphic curves possibly bordered with a Lagrangian boundary in the real context. A main consequence of these geometric calculations is the a-priori \(W^{2,2}\) estimates given in Proposition 7.4.5 for \(J\)-holomorphic curves with a Lagrangian boundary condition.

For the calculations involving the energy density we do in this section, we exploit the notion of *canonical connection* of the holomorphic tangent bundle of the almost-Kähler manifold introduced by Ehresman and Libermann (EL51) and studied by Kobayashi (Ko03). (See also (Ga97) for an extensive study of various natural connections on almost-complex manifolds.) We will see that the usage of this canonical connection on the holomorphic tangent bundle dramatically simplifies the calculations involving the derivatives of the energy density of \(J\)-holomorphic maps. Although this simplification is not essential for the elliptic estimates, we prefer to employ this optimal form of calculations, hoping that it will be useful in the future. To maximize the advantage of the canonical connection, we will adopt complexified notations for the almost-Kähler manifold \((M, \omega, J)\) and the Riemann surface \((\Sigma, j)\) in the local calculations.

### 7.1 Natural connection on almost-Kähler manifolds

In this section, we first recall the notion of *canonical connection* introduced by Ehresmann and Libermann, and studied by Gauduchon (Ga97) and Kobayashi (Ko03).
Geometric calculations

We first describe basic properties of the canonical connection on almost-Hermitian manifolds \((M, J, g)\) and then on almost-Kähler manifolds. After that, we provide the construction of the canonical connection (see Theorem 7.1.14) starting from the Levi-Civita connection of \((M, g)\) when \((M, J, g)\) is almost-Kähler, i.e., when the fundamental 2-form \(\Phi = g(J\cdot, J\cdot)\) is closed. In particular, the latter construction applies to the triple \((M, \omega, J)\), where \((M, \omega)\) is symplectic and \(J\) is compatible to \(\omega\).

### 7.1.1 Canonical connection

Let \((M, J)\) be an almost-complex manifold, i.e., \(J\) be a bundle isomorphism \(J : TM \to TM\) over the identity satisfying \(J^2 = -id\).

**Definition 7.1.1** A metric \(g\) on \((M, J)\) is called Hermitian, if \(g\) satisfies
\[
g(Ju, Jv) = g(u, v), \quad u, v \in T_x M, x \in M.
\]

We call the triple \((M, J, g)\) an almost-Hermitian manifold.

For any given almost-Hermitian manifold \((M, J, g)\), the bilinear form
\[
\Phi := g(J\cdot, \cdot)
\]
is called the fundamental two-form in \((KN96)\), which is nondegenerate.

**Definition 7.1.2** An almost-Hermitian manifold \((M, J, g)\) is an almost-Kähler manifold if the two-form \(\Phi\) above is closed.

**Definition 7.1.3** An (almost) Hermitian connection \(\nabla\) is an affine connection satisfying
\[
\nabla g = 0 = \nabla J.
\]

**Exercise 7.1.4** Prove that such a connection always exists.

In general the torsion \(T = T_\nabla\) of the almost-Hermitian connection \(\nabla\) is not zero, even when \(J\) is integrable. By definition, \(T\) has the expression
\[
T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].
\]

The following is the almost-complex version of the Chern connection in complex geometry.
7.1 Natural connection on almost-Kähler manifolds

Theorem 7.1.5 ((Ga97), (Ko03)) On any almost-Hermitian manifold \((M, J, g)\), there exists a unique Hermitian connection \(\nabla\) on \(TM\) satisfying

\[ T(X, JX) = 0 \]  \hspace{1cm} (7.1.1)

for all \(X \in TM\).

In complex geometry (Cher67) where \(J\) is integrable, a Hermitian connection satisfying (7.1.1) is called the Chern connection.

Definition 7.1.6 A canonical connection of an almost-Hermitian connection is defined to be one that has the torsion property (7.1.1).

Next we would like to derive how the almost-Kähler property \(d\Phi = 0\) for the bilinear form \(\Phi = g(\cdot, J\cdot)\) affects the torsion property of the canonical connection. For this purpose, it is best to consider the complexification of \(\nabla\). Consider the complexified tangent bundle \(T_CM = TM \otimes \mathbb{C}\) which is decomposed into

\[ T_CM = T^{(1,0)}M \oplus T^{(0,1)}M, \]

where \(T^{(1,0)}M\) (respectively \(T^{(0,1)}M\)) is the eigenspace of \(J\) with eigenvalue \(i = \sqrt{-1}\) (respectively with eigenvalue \(-i\)). The metric \(g\) extends complex-linearly to a nondegenerate complex bilinear form on \(T_CM\), still denoted by \(g\), with respect to which both \(T^{(1,0)}M\) and \(T^{(0,1)}M\) become isotropic, i.e., the form \(\Phi\) vanishes when restricted to \(T^{(1,0)}M\) and \(T^{(0,1)}M\), respectively. The associated Hermitian inner product on the complex vector space \(T^{(1,0)}M\) is then defined by

\[ \langle u, v \rangle = g(u, \overline{v}) \]  \hspace{1cm} (7.1.2)

for two complex vectors \(u, v \in T^{(1,0)}_xM\).

Now we express the complexification of such \(\nabla\) and the associated \(T\) with respect to a given frame. Choose an orthonormal frame of \(M\)

\[ \{e_1, \ldots, e_n, f_1, \ldots, f_n\} \]  \hspace{1cm} (7.1.3)

on \(TM\) so that \(f_j = Je_j\). Then the vectors

\[ u_j = \frac{1}{\sqrt{2}}(e_j - if_j), \quad j = 1, \ldots, n \]

form a unitary frame of \(T^{(1,0)}M\) and the vectors

\[ \overline{u}_j = \frac{1}{\sqrt{2}}(e_j + if_j), \quad j = 1, \ldots, n \]
form a unitary frame of $T^{(0,1)}M$. Let

$$\{\alpha^1, \ldots, \alpha^n, \beta^1, \ldots, \beta^n\} \quad (7.1.4)$$

be the dual frame of $(7.1.3)$. The complex-valued 1-forms

$$\theta^j = \frac{1}{\sqrt{2}}(\alpha^j + i\beta^j), \quad j = 1, \ldots, n$$

form a unitary frame of $\Lambda^{(0,1)}(M) = (T^{(1,0)}M)^*$ that is dual to the unitary frame $\{u_1, \ldots, u_n\}$.

Any almost-Hermitian connection $\nabla$ on $TM$ induces a connection of the complex vector bundle $\Lambda^{(1,0)}M$, which is defined by a locally defined, matrix-valued complex 1-form $\omega = (\omega^i_j)_{i,j=1,2,\ldots,n}$ that is skew-Hermitian, i.e., $\overline{\omega^i_j} = -\omega^j_i$. The corresponding torsion, denoted by $\Theta$, is a skew-Hermitian two-form with values in $T^{(1,0)}M$ and has the expression

$$\Theta = \Pi T_C, \quad (7.1.5)$$

where $T_C = T \otimes \mathbb{C}$ is the complex linear extension of $T$ and $\Pi$ is the projection to $T^{(1,0)}M$.

The first structure equation of this unitary connection is given by

$$d\theta^i = -\sum \omega^i_j \wedge \theta^j + \Theta^i \quad (7.1.6)$$

or simply as $d\theta = -\omega \wedge \theta + \Theta$ as a matrix equation.

The second structure equation is given by

$$d\omega^i_j = -\sum \omega^i_k \wedge \omega^k_j + \Omega^i_j \quad (7.1.7)$$

or simply as $d\omega = -\omega \wedge \omega + \Omega$, where $\Omega$ is the curvature two-form. The two decompose into

$$\Theta = \Theta^{(2,0)} + \Theta^{(1,1)} + \Theta^{(0,2)},$$

$$\Omega = \Omega^{(2,0)} + \Omega^{(1,1)} + \Omega^{(0,2)}.$$

**Theorem 7.1.7**  The $(0, 2)$-component $\Theta^{(0,2)}$ of the torsion form $\Theta$ is independent of the choice of almost-Hermitian connection.

**Exercise 7.1.8**  Prove this theorem.

The $(0, 2)$-form $\Theta^{(0,2)}$ is called the **torsion** of the almost-complex structure $J$. The integrability of $J$ is given by $\Theta^{(0,2)} = 0$. On the other hand, the components $\Theta^{(1,1)}$ and $\Theta^{(2,0)}$ depend on the choice of connection.

The following is the definition of canonical connection in complexified form which is equivalent to Definition 7.1.6.
Definition 7.1.9 Let \((M, J, g)\) be an almost-Hermitian manifold. A connection \(\nabla\) of \(TM\) is the canonical connection, if the torsion form \(\Theta\) on the complex vector bundle \(T^{(1,0)}M\) satisfies \(\Theta^{(1,1)} = 0\).

It was proved by Kobayashi that, if \((M, J, g)\) is almost-Kähler, then the following holds.

Theorem 7.1.10 (Kobayashi, (Ko03)) Let \((M, J, g)\) be almost-Kähler and let \(\nabla\) be the canonical connection of \(TM\) and its complexification. Then \(\Theta^{(2,0)} = 0\) in addition, and hence \(\Theta\) is of type \((0, 2)\), and satisfies

\[
\sum \Theta^i \wedge \bar{\theta}^i = 0. \quad (7.1.8)
\]

Equivalently, we have the following real description of \(\Theta\) being of type \((0, 2)\).

Proposition 7.1.11 Let \((M, J, g)\) be an almost-Kähler manifold and let \(\nabla\) be its canonical connection. Then the torsion tensor \(T\) satisfies

\[
T(JY, Z) = T(Y, JZ), \quad JT(JY, Z) = T(Y, Z)
\]

for all vector fields \(Y, Z\).

Remark 7.1.12 When we start with a general affine connection on the real tangent bundle \(TM\), there will be many different such affine connections whose complexification leads to the connection on the holomorphic tangent bundle on \(T^{(1,0)}M\) with \(\Theta^{(1,1)} = 0\). Such an affine connection is not unique.

In the next section, we show that the Levi-Civita connection is another such choice besides the above-mentioned canonical Hermitian connection. This Levi-Civita connection is, however, not \(J\)-linear. Because of this, we prefer to use the unique Hermitian connection given by the above theorem.

7.1.2 Levi-Civita connection

Recall that, on considering an almost-Hermitian manifold \((M, J, g)\) as a Riemannian manifold, it induces the Levi-Civita connection on \(TM\), which can be extended to \(T\mathbb{C}M\) complex linearly.

Now we complex linearly extend this Levi-Civita connection to \(T\mathbb{C}M\) and express it in terms of the above-mentioned moving frame. The Levi-Civita connection \(\nabla\) on \(T\mathbb{C}M\) in terms of the basis
\{u_1, \ldots, u_n, \overline{u}_1, \ldots, \overline{u}_n\}

is locally expressed by

$$\nabla u_j = \sum_i \lambda^i_j \otimes u_i + \sum_\ell \mu^\ell_j \otimes \overline{u}_\ell$$

and their complex conjugates. The first structure equation becomes

$$d \left( \frac{\partial}{\partial \vartheta} \right) = -\left( \frac{\lambda}{\mu}, -\frac{\mu}{\lambda} \right) \wedge \left( \frac{\partial}{\partial \vartheta} \right) \wedge \left( \frac{\partial}{\partial \overline{\vartheta}} \right).$$

(7.1.9)

Since $\overline{\lambda} = -\lambda^t$, the skew-Hermitian matrix $\lambda = (\lambda^i_j)$ induces a connection on $T^{(1,0)}M$, the so-called Lichnerowitz connection. The first structure equation for $\lambda$ is given by

$$d \vartheta^j = -\sum_i \lambda^i_j \wedge \vartheta^i - \sum_i \mu^i_j \wedge \overline{\vartheta}^i.$$

Therefore, on recalling the definition (7.1.6) of the torsion form, the torsion form $\Theta'$ of this connection on $T^{(1,0)}M$ is given by

$$(\Theta')^j = -\sum_i \mu^i_j \wedge \overline{\vartheta}^i.$$  

(7.1.10)

We write

$$\mu^i_j = \sum_{i, \ell} N^j_{\ell i} \overline{\vartheta}^\ell + \sum_{i, k} N^j_{i k} \vartheta^k.$$ 

The following theorem by Kobayashi clarifies the relations between the canonical connection on $T^{(1,0)}M$ of $(M, J, g)$ and the Levi-Civita connection on $(M, g)$.

**Theorem 7.1.13** (Kobayashi) Suppose $(M, J, g)$ is almost-Kähler. Then the connection on $T^{(1,0)}M$ induced by the canonical connection in the previous section coincides with the Lichnerowitz connection, i.e., $\omega^i_j = \lambda^i_j$. In particular, we have $\Theta = \Theta'$ and so

$$N^j_{k i} = 0, \quad N^j_{i i} = N^j_{i i}.$$ 

The $(1, 2)$-tensor associated with $N^j_{i i}$ is nothing but the complexified Nijenhuis torsion tensor of the almost-complex structure $J$. If we set $N^j_{\overline{i} i} = N^j_{j i}$, then it is easy to check that (7.1.8) implies the following symmetry property:
\[ N_{j\bar{i}} = N_{\bar{i}j} = N_{ij\bar{t}}. \]  
\[ (7.1.11) \]

We substitute (7.1.10) and
\[ \omega^j_i = \sum_{\ell} \omega^j_{i\ell} \bar{\theta}^\ell + \sum_{\ell} \omega^j_{i\bar{\ell}} \bar{\theta}^\ell + \sum_{i,\ell} N^j_{i\ell} \bar{\theta}^\ell \]
into (7.1.6), and write
\[ d\theta^j = -\sum_{i,\ell} \omega^j_{i\ell} \theta^\ell \wedge \bar{\theta}^i - \sum_{i,\ell} \omega^j_{i\bar{\ell}} \bar{\theta}^\ell \wedge \theta^i + \sum_{i,\ell} N^j_{i\ell} \bar{\theta}^\ell \wedge \bar{\theta}^i. \]

Finally we examine the relationship between the Levi-Civita connection and the canonical connection on almost-Kähler manifolds. Let \( \nabla^{LC} \) be the Levi-Civita connection. Consider the standard averaged connection \( \nabla^{av} \) of multiplication \( J : TM \to TM \),
\[ \nabla^{av}_X Y := \frac{\nabla^{LC}_X Y + J^{-1} \nabla^{LC}_X (JY)}{2} = \nabla^{LC}_X Y - \frac{1}{2} J(\nabla^{LC}_X J)Y. \]  
\[ (7.1.12) \]

We then have the following proposition stating that this connection becomes the canonical connection. Its proof can be found in (KN96, Theorem 3.4) or from section 2 (Ga97), especially (2.2.10) (Ga97).

**Theorem 7.1.14** Assume that \( (M, g, J) \) is almost-Kähler, i.e., the two-form \( \omega = g(J \cdot, \cdot) \) is closed. Then the average connection \( \nabla^{av} \) defines the canonical connection of \( (M, g, J) \), i.e., the connection is \( J \)-linear and preserves the metric, and its complexified torsion is of type \((0, 2)\).

**Proof** Using the definition (7.1.12), we compute the torsion \( T \) of \( \nabla^{av} \) and get
\[ T(X, Y) = \nabla^{av}_X Y - \nabla^{av}_Y X - [X, Y] \]
\[ = T^{LC}(X, Y) - \frac{1}{2} J(\nabla^{LC}_X J)Y + \frac{1}{2} J(\nabla^{LC}_Y J)X \]
\[ = -\frac{1}{2} J(\nabla^{LC}_X J)Y + \frac{1}{2} J(\nabla^{LC}_Y J)X, \]  
\[ (7.1.13) \]
where we use the fact that \( T^{LC}(X, Y) = \nabla^{LC}_X Y - \nabla^{LC}_Y X - [X, Y] = 0 \) for the second equality. On the other hand, \( d\omega = 0 \) implies \( \nabla^{LC}_X J + J(\nabla^{LC}_X J) = 0 \). (See (2.2.10) (Ga97).) Substituting this into (7.1.13) with \( Y = JX \), we obtain \( T(JX, X) = 0 \), which is the defining property of the canonical connection. Then Proposition 7.1.11 implies that the torsion is of type \((0, 2)\). \( \square \)
7.1.3 On symplectic manifolds \((M, \omega)\)

Now we consider a symplectic manifold \((M, \omega)\).

**Definition 7.1.15** An almost-complex structure \(J\) is called *compatible* with \(\omega\) if it satisfies the following:

(i) *(J-positive)* \(\omega(u, Ju) \geq 0\) and equality holds if and only if \(u = 0\)
(ii) *(J-Hermitian)* \(\omega(Ju, Jv) = \omega(u, v)\).

We denote by \(\mathcal{J}_\omega\) the set of such almost-complex structures.

A polar decomposition guarantees that there always exists a compatible almost-complex structure on any symplectic manifold \((M, \omega)\).

**Exercise 7.1.16** Prove this statement.

In fact, we have the following general lemma by Gromov (Gr85).

**Lemma 7.1.17** *(Gromov)* The set \(\mathcal{J}_\omega\) is a nonempty contractible infinite-dimensional (Fréchet) manifold.

For any \(J\) compatible with \(\omega\), the bilinear form

\[ g = \omega(\cdot, J\cdot) \]

defines a Hermitian metric with respect to \(J\) and hence the triple \((M, J, g)\) defines an *almost-Kähler manifold* in the sense of the above definition. By definition, the fundamental two-form of this metric \(g\) becomes \(\Phi = \omega\). We denote this almost-Kähler manifold by \((M, \omega, J)\) highlighting the symplectic form \(\omega\).

From now on, we will mostly use the canonical almost-Hermitian connection for our calculations provided in Theorem 7.1.5, instead of the Levi-Civita connection on \(TM\) of the almost-Kähler manifold \((M, \omega, J)\), unless otherwise said. However, we remark that by Theorem 7.1.5 both connections induce the same associated canonical connection on the holomorphic tangent bundle \(T^{(1,0)}M\).

**Example 7.1.18** Consider the standard Euclidean symplectic vector space

\[ \mathbb{R}^{2n}, \quad \omega_0 = \sum_{j=1}^{n} dq_j \wedge dp_j \]
with the isomorphism
\[ \mathbb{R}^{2n} \to \mathbb{C}^n ; (q^1, \ldots, q^n, p_1, \ldots, p_n) \mapsto (z_1, \ldots, z_n), \quad z_j = q^j + \sqrt{-1}p_j. \]
It follows that the Euclidean inner product \( g \) is written as \( g = \omega_0(\cdot, J_0 \cdot) \) with respect to \( J_0 \) induced from the complex multiplication by \( i = \sqrt{-1} \) on \( \mathbb{C}^n \), i.e., \( J_0 \) is the endomorphism determined by
\[
J_0 \left( \frac{\partial}{\partial q^j} \right) = \frac{\partial}{\partial p_j}, \quad J_0 \left( \frac{\partial}{\partial p^j} \right) = -\frac{\partial}{\partial q^j}
\]
for \( j = 1, \ldots, n \). Note that \( \mathbb{R}^{2n} \cong \mathbb{C}^n \) carries three canonical bilinear forms: the symplectic form \( \omega_0 \), the Euclidean inner product \( g \) and the Hermitian inner product \( \langle \cdot, \cdot \rangle \). Our convention for the relation among these three is
\[
\langle \cdot, \cdot \rangle = g(\cdot, \cdot) - i\omega_0(\cdot, \cdot).
\]
In other words, the Hermitian inner product is complex linear in the first argument and complex anti-linear in the second argument.

### 7.2 Global properties of J-holomorphic curves

An ingenious idea of Gromov introduced in (Gr85) that revolutionized the area of symplectic topology is the idea of studying the space of almost-complex maps
\[ f : (\Sigma, j) \to (M, J), \]
where \((\Sigma, j)\) is another almost-complex manifold.

**Definition 7.2.1** An almost-complex map \( f : (\Sigma, j) \to (M, J) \) is called a \((j, J)\)-holomorphic map.

However, unless both \( j \) and \( J \) are integrable, not even a local existence theorem of such maps is possible when \( \dim \Sigma > 2 \). On the other hand, when \( \dim_{\mathbb{R}} \Sigma = 2 \), there is the following local existence theorem. Very often, we omit \( j \) or call the pair \((j, f)\) a \( J \)-holomorphic map, and its image an (unparameterized) \( J \)-holomorphic curve.

**Theorem 7.2.2** (Nijenhuis–Wolf (NiW63)) At each point \( p \in M \) and for each complex tangent 2-plane \( \tau_p \) at \( x \), there exists \( \epsilon > 0 \) and a \( J \)-holomorphic disc \( f : (D^2(\epsilon), j) \to (M, J) \) such that \( f(0) = p \) and \( \tau_p = \text{Image } T_0f \).
The proof of this theorem relies on the elliptic theory of nonlinear partial differential equations, noting that the almost-complex condition $df \circ j = J \circ df$ gives rise to a quasi-linear first-order elliptic partial differential equation. In fact, for the case $(D^2(\epsilon), j)$, this is equivalent to

$$\frac{\partial f}{\partial x} + J(f) \frac{\partial f}{\partial y} = 0,$$

where $\partial f/\partial x := df(\partial/\partial x)$, $\partial f/\partial y := df(\partial/\partial y)$ and $(x, y)$ is the standard coordinate of $D^2(\epsilon)$. Of course, when $J$ is integrable, this equation becomes exactly the equation for holomorphic curves in complex coordinates $(z_1, \ldots, z_n)$ of $M$ and $z = x + \sqrt{-1} y$ on $\Sigma$, if we identify $J$ with the complex multiplication by $\sqrt{-1}$.

Now, we study some global properties of $J$-holomorphic curves. For the purpose of maintaining consistency with the notations we use for the later calculations, we will switch from the notation $Tu$ to $du$ and regard it as a vector-valued one-form with its values lying in $u^* TM$. Then we have a decomposition

$$du = \partial_J u + \overline{\partial_J} u,$$

(7.2.14)

where $\partial_J u$ (respectively $\overline{\partial_J} u$) is the complex linear (respectively anti-complex linear) part of $du$ with respect to $(j, J)$:

$$\partial_J u := \frac{1}{2} (du - J \cdot du \cdot j),$$

(7.2.15)

$$\overline{\partial_J} u := \frac{1}{2} (du + J \cdot du \cdot j).$$

(7.2.16)

Then $u$ is $J$-holomorphic if and only if $\overline{\partial_J} u = 0$.

Fix a Hermitian metric $h$ on $(\Sigma, j)$. The norm $|du|$ of the map

$$du : (T \Sigma, h) \to (TM, g)$$

with respect to any Riemannian metric $g$ is defined by

$$|du|_g^2 := \sum_{i=1}^{2} |du(e_i)|_g^2,$$

where $\{e_1, e_2\}$ is an orthonormal frame of $T \Sigma$ with respect to $h$. This definition does not depend on the choice of the orthonormal frame $\{e_1, e_2\}$. In coordinates, $(x^1, x^2)$ on $\Sigma$ and $(y_1, y_2, \ldots, y_{2n})$ on $M$, we write $g = \sum g_{\alpha \beta} dy^\alpha dy^\beta$ and $h = \sum_{i,j} h_{ij} dx^i dx^j$ and $(h^{ij}) = (h_{ij})^{-1}$. Then $|du|_g^2$, called the harmonic energy density function, is given by the expression

$$|du|_g^2 = \sum_{i,j,\alpha,\beta} g_{\alpha \beta}(u(x)) h^{ij}(x) \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j}.$$
The following are consequences from the definition of a $J$-holomorphic map and the compatibility of $J$ with $\omega$.

**Proposition 7.2.3** Denote $g_J = \omega(\cdot, J\cdot)$ and the associated norm by $|\cdot| = |\cdot|_J$. Fix a Hermitian metric $h$ of $(\Sigma, j)$, and consider a smooth map $u : \Sigma \to M$. Then we have

1. $|du|^2 = |\bar{\partial}Ju|^2 + |\partial J u|^2$
2. $2u^*\omega = (-|\bar{\partial}Ju|^2 + |\partial J u|^2)dA$, where $dA$ is the area form of the metric $h$ on $\Sigma$.

**Proof** Since the definition of the norm $|du|^2$ given above does not depend on the choice of the orthonormal frame $\{e_1, e_2\}$, we choose an orthonormal local frame of $T\Sigma$ with respect to $h$ such that $je_1 = e_2$ and denote by $\{\alpha_1, \alpha_2\}$ its dual frame.

By definition,

$$|du|^2 = |du(e_1)|^2 + |du(e_2)|^2$$

and

$$|\bar{\partial}Ju|^2 = |\bar{\partial}Ju(e_1)|^2 + |\bar{\partial}Ju(e_2)|^2,$$

$$|\partial J u|^2 = |\partial J u(e_1)|^2 + |\partial J u(e_2)|^2.$$

But by the choice $e_2 = je_1$,

$$\bar{\partial}Ju(e_1) = \frac{du(e_1) + J\, du(e_2)}{2}, \quad \bar{\partial}Ju(e_2) = \frac{du(e_2) - J\, du(e_1)}{2}$$

and

$$\partial J u(e_1) = \frac{du(e_1) - J\, du(e_2)}{2}, \quad \partial J u(e_2) = \frac{du(e_2) + J\, du(e_1)}{2}.$$

The identity (1) follows immediately from a straightforward calculation using the fact that the endomorphisms $J$ and $j$ are orthogonal with respect to the metrics $g$ and $h$, respectively.

For the proof of the identity (2), we express the area form $dA$ as

$$dA = \alpha_1 \wedge \alpha_2.$$ 

Then, from the compatibility of $J$ with $\omega$, a straightforward calculation shows that

$$|\partial J u|^2 - |\bar{\partial}Ju|^2 = -2(du(e_1), J\, du(e_2)) = 2(J\, du(e_1), du(e_2))$$

$$= 2\omega(du(e_1), du(e_2)) = 2u^*\omega(e_1, e_2).$$

This implies (2). Hence the proof. □
From these identities, we obtain the following \textit{calibrated} property of $J$-holomorphic curves with respect to the symplectic form $\omega$.

\textbf{Corollary 7.2.4} \textit{For any smooth map $u : \Sigma \to M$, we have}

$$\frac{1}{2} \int_{\Sigma} |du|^2_J \, dA \geq \int u^* \omega$$

\textit{and equality holds precisely when $u$ is $J$-holomorphic.}

Note that when $\Sigma$ is a closed surface without a boundary $\int u^* \omega$ is constant in a fixed homology class. Therefore, any $J$-holomorphic curve minimizes the \textit{harmonic energy} $\frac{1}{2} \int_{\Sigma} |du|^2_J \, dA$ in a fixed homology class. This immediately gives rise to the following.

\textbf{Proposition 7.2.5} \textit{When $u : (\Sigma, j) \to (M, J)$ is $J$-holomorphic, we have}

$$\text{Area}_{g_J} u = \frac{1}{2} \int |du|^2_J = \int u^* \omega.$$

Therefore $\text{Area}_{g_J} u$ depends only on the homology class represented by $u$. In particular, for any $J$-holomorphic map $u : (\Sigma, j) \to (P, J)$ representing a fixed homology class $A = [u]$,

$$\text{Area}_{g_J}(u) = [\omega](A). \quad (7.2.17)$$

\textbf{Corollary 7.2.6} \textit{Near each regular point of $u$ on $\Sigma$, the image of a $J$-holomorphic map is a minimal surface with respect to the metric $g_J = \omega(\cdot, J \cdot)$.}

\textbf{Remark 7.2.7} \textit{In applications, one must vary the almost-complex structure and consider a family of $J$ contained in a compact subset $K$ of $\mathcal{J}_\omega$. However, when we make estimates and develop convergence arguments of a sequence $\{f_i\}$ of maps, we have to use the given fixed metric $g$ among compatible metrics. Then (7.2.17) should be replaced by}

$$\text{Area}_g(f) \leq C(A, K)$$

\textit{for any $J$-holomorphic map $f$ and $J \in K \subset \mathcal{J}_\omega$, where $C(A, K)$ is a constant depending only on $A$ and $K$. This uniform area estimate will be an important step for obtaining a uniform $C^1$ estimate and hence for the study of compactness properties of $J$-holomorphic maps (for varying $J$ in a compact set).}
7.3 Calculations of $\Delta e(u)$ on shell

We will carry out some geometric calculations involving the harmonic energy density function

$$e(u) := |du|^2$$

for a $J$-holomorphic map. This will lead to the elliptic estimates for solutions of the Cauchy–Riemann equation that are needed for more detailed study of the pseudoholomorphic curves.

To maximize the advantage of the canonical connection, we will adopt complexified notations for the almost-Kähler manifold $(M, \omega, J)$ and the Riemann surface $(\Sigma, j)$ in the calculations below.

This being said, let $(\Sigma, j)$ be an open Riemann surface that is equipped with a Hermitian metric $h$ compatible with the given complex structure $j$. The complexification $T_C M = TM \otimes \mathbb{C}$ has the decomposition

$$T_C \Sigma = T^{(1,0)} \Sigma \oplus T^{(0,1)} \Sigma, \quad T_C M = T^{(1,0)} M \oplus T^{(0,1)} M.$$  (7.3.18)

Upon extending the derivative $du$ complex linearly to a bundle map,

$$du : T_C \Sigma \to T_C M,$$

it can be expressed as the natural block diagonal matrix

$$du = \begin{pmatrix} (\partial u)' & (\partial u)'' \\ (\bar{\partial} u)' & (\bar{\partial} u)'' \end{pmatrix},$$  (7.3.19)

where $(\cdot)'$ is the $(1,0)$ part and $(\cdot)''$ the $(0,1)$ part of $(\cdot) \in T_C M$ in its image. Here

$$\partial u \in \Omega^{(1,0)}(\Sigma; u^* T_C M), \quad \bar{\partial} u \in \Omega^{(0,1)}(\Sigma; u^* T_C M)$$  (7.3.20)

are the $(1,0)$ and $(0,1)$ components of $du \in \Omega^1(\Sigma; u^* T_C M)$ as $T_C M$-valued one-forms on $\Sigma$, respectively. Here we consider the tangent map $du : T_C \Sigma \to T_C M$ as a one-form.

On the other hand, we recall that the real operator $du$ has the decomposition

$$du = \partial_J u + \bar{\partial}_J u.$$  

Similarly we extend $\partial_J u$ and $\bar{\partial}_J u$ complex linearly to $T_C \Sigma \to T_C M$.

**Lemma 7.3.1** In complexification $T_C M$ of the tangent bundle $T \Sigma$, $\partial_J u, \bar{\partial}_J u : T_C \Sigma \to T_C M$ has the form

$$\partial_J u = \begin{pmatrix} (\partial u)' & 0 \\ 0 & (\bar{\partial} u)'' \end{pmatrix}, \quad \bar{\partial}_J u = \begin{pmatrix} 0 & (\partial u)'' \\ (\bar{\partial} u)' & 0 \end{pmatrix}.$$
as $u^*T_\Sigma M$-valued one-forms on $\Sigma$. In particular, $\partial_J u$ preserves the type of the vectors.

**Proof** We can write

$$du(\xi) = \left(\frac{du(\xi) - iJ \cdot du(\xi)}{2}\right) + \left(\frac{du(\xi) + iJ \cdot du(\xi)}{2}\right)$$

for all $\xi \in T_\Sigma$, which respects the decomposition $T_\Sigma M = T^{(1,0)}M \oplus T^{(0,1)}M$.

Now let $\xi \in T^{(1,0)} \Sigma$ and evaluate

$$\partial_J u(\xi) = \frac{du(\xi) - J \cdot du(\xi)}{2} = \frac{du(\xi) - iJ \cdot du(\xi)}{2}, \quad (7.3.21)$$

where we use $j\xi = i\xi$. The right-hand side is precisely the projection of $du(\xi)$ to $T^{(1,0)}M$. We compute

$$J \left(\frac{du(\xi) - iJ \cdot du(\xi)}{2}\right) = i \left(\frac{du(\xi) - iJ \cdot du(\xi)}{2}\right).$$

On the other hand, for $\xi \in T^{(0,1)} \Sigma$, we have

$$\partial_J u(\xi) = \frac{du(\xi) - J \cdot du(\xi)}{2} = \frac{du(\xi) + iJ \cdot du(\xi)}{2} \in T^{(0,1)}M,$$

which is the $T^{(0,1)}M$ projection of $du(\xi)$. It follows from this that

$$\partial_J u = \begin{pmatrix} (\partial u)' & 0 \\ 0 & (\partial u)'' \end{pmatrix}. \quad \square$$

We obtain the formula for $\partial_J u$ similarly. This finishes the proof.

$\Omega^1(u^*T_\Sigma M)$ is decomposed into the sum

$$\Omega^1(u^*T_\Sigma M) = \Omega^{(0,1)}(u^*T_\Sigma M) \oplus \Omega^{(0,1)}(u^*T_\Sigma M)$$

and, since we are on the Riemann surface, we have

$$\Omega^{(2,0)}(u^*T_\Sigma M) = 0 = \Omega^{(0,2)}(u^*T_\Sigma M)$$

and so

$$\Omega^{(1,1)}(u^*T_\Sigma M).$$

Consider the pull-back connection, again denoted by $\nabla$ on $u^*TM$. Combined with the Hermitian connection of $(\Sigma, j, h)$, it induces a connection on $T^* \Sigma \otimes u^*TM$, which we again denote by $\nabla$. We denote by $d^\nabla$ the skew-symmetrization of the covariant derivative $\nabla$ given by

$$d^\nabla(\alpha)(\xi_1, \xi_2) = (\nabla_{\xi_1} \alpha)(\xi_2) - (\nabla_{\xi_2} \alpha)(\xi_1),$$
which defines an operator
\[ d^\nabla : \Omega^1(u^*TM) \to \Omega^2(u^*TM). \]

Considering \( \alpha = du \) as a one-form in \( \Omega^1(u^*TM) \), we have the following lemma.

**Lemma 7.3.2** As a two-form with values in \( u^*TM \), we have
\[ d^\nabla (du) = u^*T, \] (7.3.22)
where \( T \) is the torsion tensor of \( \nabla \).

**Proof** For given \( \xi_1, \xi_2 \in \Gamma(T\Sigma) \), we evaluate
\[
d^\nabla (du)(\xi_1, \xi_2) = \nabla_{\xi_1}(du)(\xi_2) - \nabla_{\xi_2}(du)(\xi_1)
= \nabla_{du(\xi_1)}(du(\xi_2)) - du(\nabla_{\xi_1}(\xi_2)) - (\nabla_{du(\xi_2)}(du(\xi_1)) - du(\nabla_{\xi_2}(\xi_1))
= \nabla_{du(\xi_1)}(du(\xi_2)) - \nabla_{du(\xi_2)}(du(\xi_1)) - du([\xi_1, \xi_2])
= T(du(\xi_1), du(\xi_2)) = u^*T(\xi_1, \xi_2),
\]
which finishes the proof. \( \Box \)

We extend the operator \( du \) and \( T \) complex linearly on the complexifications \( T_C\Sigma \) and \( T_C\Sigma \).

**Corollary 7.3.3** Suppose that \( u \) is J-holomorphic, i.e., \( \bar{\partial}_J u = 0 \). Then \( u^*T = 0 \) and so \( d^\nabla (du) = 0 \).

**Proof** We evaluate
\[
u^*T(\eta, j\eta) = T(du(\eta), du(j\eta)) = T(du(\eta), Jdu(\eta)) = 0.
\]
Here we use \( du \circ j = J \circ du \) for the second equality and Theorem 7.1.5 for the third equality. This finishes the proof. \( \Box \)

Using Corollary 7.3.3, we obtain the following.

**Theorem 7.3.4** Let \( u \) be J-holomorphic. Then \( \partial u \) is harmonic, i.e., satisfies \( \Delta(\partial u) = 0 \).

**Proof** Let \( * \) be the Hodge star operator on \( \Sigma \). Then \( *\xi = -i\xi \) for \( \xi \in (T^*\Sigma)^{1,0} \). Therefore
\[
* \partial u = -i \partial u.
\] (7.3.23)
Using the formulae for the adjoint
\[(d^\nabla)^* = -*d^\nabla^\ast\]
and applying them to \(\partial u\), we obtain
\[(d^\nabla)^*(\partial u) = -*d^\nabla^\ast(\partial u) = i* d^\nabla(\partial u).\]
This vanishes by Corollary 7.3.3 since \(d^\nabla \partial u = d^\nabla du|_{T^{(1,0)}\Sigma}\).

Now we consider the Hodge Laplacian
\[\Delta = \Delta^u := d^\nabla(d^\nabla)^\ast + (d^\nabla)^\ast d^\nabla\]
and compute
\[\Delta(\partial u) = d^\nabla(d^\nabla)^\ast(\partial u) + (d^\nabla)^\ast d^\nabla(\partial u) = 0.\] (7.3.24)
This finishes the proof. \(\square\)

We now recall the Bochner–Weitzenböck formula
\[\Delta(\partial u) = \nabla^* \nabla(\partial u) + K(\partial u) + R(\partial u, \partial u)\partial u\] (7.3.25)
where \(\nabla^* \nabla\) is the trace Laplacian, \(K\) is the Gauss curvature of \((\Sigma, h)\) and \(R\) is
the sectional curvature of the connection \(\nabla\) of \(M\), extended to \(T_{\mathbb{C}}M\) complex
linearly. (See Appendix A.1 for a complete derivation of the Weitzenböck formula for vector-valued differential forms on general Riemannian manifolds.)

Combining Theorem 7.3.4 and (7.3.25), we have derived
\[\nabla^* \nabla(\partial u) = \Delta(\partial u) - K \partial u - R(\partial u, \partial u)\partial u = -K \partial u - R(\partial u, \partial u)\partial u.\] (7.3.26)

We compute the Laplacian of the energy density \(e(u)\) for a \(J\)-holomorphic map
\(u : (\Sigma, j) \rightarrow (M, J)\) where \(\Sigma\) may have nonempty boundary \(\partial \Sigma\). Recall that we have the identity
\[e(u) = |du|^2 = |\partial u|^2 + |\bar{\partial} u|^2 = 2|\partial u|^2\]
for a \(J\)-holomorphic map. In other words, the energy density can be written purely in terms of the \((1,0)\)-form \(\partial u\) with values in \(T^{(1,0)}M\). Therefore we compute \(\Delta|\partial u|^2\) instead of \(\Delta|du|^2\) using this identity.

Observing that \(du\) is a real operator, we derive
\[
\frac{1}{2} \Delta e(u) = \Delta(\partial u, \partial u) = \langle \nabla^* \nabla \partial u, \partial u \rangle - 2|\nabla \partial u|^2 + \langle \partial u, \nabla^* \nabla \partial u \rangle
= \langle -2K \partial u, \partial u \rangle - 2\langle R(\partial u, \partial u)\partial u, \partial u \rangle - 2|\nabla \partial u|^2.
\] (7.3.27)

Hence
\[|\nabla du|^2 = 2|\nabla \partial u|^2 = -\frac{1}{2} \Delta e(u) - 2\langle K \partial u, \partial u \rangle - 2\langle R(\partial u, \partial u)\partial u, \partial u \rangle.\] (7.3.28)
7.4 Boundary conditions

Here we apply the following lemma for the first equality.

**Lemma 7.3.5**  We have $2|\nabla \partial u|^2 = |\nabla u|^2$.

**Proof**  We write $\nabla u = \nabla (\partial u + \overline{\partial u})$. Let $\{e_1, e_2\}$ be an orthonormal frame of $T\Sigma$. Then we have

$$|\nabla (\partial u)|^2 = |\nabla_{e_1} (\partial u)|^2 + |\nabla_{e_2} (\partial u)|^2.$$  

On the other hand, by the definition $\partial u = du|_{T^{1,0}\Sigma}$,

$$|\nabla_{e_1} (\partial u)|^2 = |\nabla_{e_1} (du)(\xi)|^2$$  

for $\xi = (1/\sqrt{2})(e_1 - ie_2)$. Similarly, we have

$$|\nabla_{e_1} (\overline{\partial u})|^2 = |\nabla_{e_1} (du)(\overline{\xi})|^2.$$  

But we have

$$\nabla_{e_1} (du)(\overline{\xi}) = \nabla_{e_1} (du)(\overline{\xi}) - du(\nabla_{e_1} (\overline{\xi})) = \nabla_{e_1} (du)(\overline{\xi})$$

and hence $|\nabla_{e_1} (\overline{\partial u})|^2 = |\nabla_{e_1} (\partial u)|^2$. Similarly, we can also obtain $|\nabla_{e_2} (\overline{\partial u})|^2 = |\nabla_{e_2} (\partial u)|^2$. This proves that $|\nabla (\partial u)|^2 = |\nabla \overline{\partial u}|^2$. Recalling that $|\nabla du|^2 = |\nabla u|^2 + |\nabla \overline{\partial u}|^2$, we have finished the proof. \(\square\)

For the case in which $\partial \Sigma = \emptyset$, integration of (7.3.28) over $\Sigma$ provides

$$\int_{\Sigma} |\nabla du|^2 = -2 \int_{\Sigma} \langle K \partial u, \partial u \rangle - 2 \int_{\Sigma} \langle R(du, du) \partial u, \partial u \rangle. \tag{7.3.29}$$

**Proposition 7.3.6**  For any $J$-holomorphic map $u : \Sigma \to M$ with $\partial \Sigma = \emptyset$, we have the inequality

$$\int_{\Sigma} |\nabla du|^2 \leq \int_{\Sigma} \max_{z \in \Sigma} (-K)(z)|du|^2 + \frac{1}{2} \int_{\Sigma} \max_{x \in M} |R(x)| |du|^4. \tag{7.3.30}$$

**Proof**  This follows from (7.4.31) and the identity $|du|^2 = 2|\partial u|^2$. \(\square\)

### 7.4 Boundary conditions

When $\partial \Sigma \neq \emptyset$, integration of (7.3.28) over $\Sigma$ gives rise to

$$\int_{\Sigma} |\nabla du|^2 = - \int_{\partial \Sigma} \frac{1}{2} \frac{\partial}{\partial v} \langle du, du \rangle - 2 \int_{\Sigma} \langle K \partial u, \partial u \rangle$$

$$- 2 \int_{\Sigma} \langle R(du, du) \partial u, \partial u \rangle. \tag{7.4.31}$$

Therefore we need to take care of the boundary contribution.
Suppose that $Y \subset M$ is a submanifold and denote its second fundamental form by $B^\nabla : \text{Sym}^2(TY) \to NY$ of the canonical connection $\nabla$. Since we are not using the Levi-Civita connection, we introduce the definition of the second fundamental form for the Riemannian connection, which is not necessarily torsion-free in general.

**Definition 7.4.1** Let $\nabla$ be any Riemannian connection, and not necessarily torsion-free. The second fundamental form $B^\nabla : TR \times TR \to NR$ is characterized by the equality

$$\langle B^\nabla(X, Y), \nu \rangle = -\frac{1}{2}\langle \nabla_X Y + \nabla_Y X, \nu \rangle$$

for any $X, Y \in TR$ and a normal vector $\nu \in NR$.

**Exercise 7.4.2** Prove that the above definition defines a symmetric 2-tensor $B^\nabla$ with values in $NR$.

By the definition of the second fundamental form $B^\nabla$ of $Y$, we can rewrite

$$\frac{1}{2} \frac{\partial}{\partial \nu} \langle du, du \rangle = \left(\langle du, D_\nu (du) \rangle \right) = -\left(\langle B^\nabla(du, du), \frac{\partial u}{\partial \nu} \rangle \right) + \left( T \left( \frac{\partial}{\partial \nu}, du \right), du \right)$$

if $\partial u/\partial \nu \perp TY$. However, on $\partial \Sigma$, we compute

$$T \left( \frac{\partial}{\partial \nu}, du \right) = T \left( \frac{\partial u}{\partial \nu}, \frac{\partial u}{\partial t} \right) dt = -T \left( J \frac{\partial u}{\partial t}, \frac{\partial u}{\partial t} \right) dt = 0,$$

where the vanishing again arises from the general property of the canonical connection stated in Theorem 7.1.5.

Combining the above calculations, we obtain the following a-priori inequality for the case with a boundary, which immediately follows from (7.3.28).

**Proposition 7.4.3** For any $J$-holomorphic map $u$ satisfying the Neumann boundary condition, i.e., satisfying $\partial u/\partial \nu \in NY$, we have the inequality

$$\int_{\Sigma} |\nabla du|^2 \leq \int_{\partial \Sigma} \max_{y \in Y} |B^\nabla||du|^3 + \int_{\Sigma} \max_{z \in \Sigma} (-K(z))|du|^2 + \frac{1}{2} \int_{\Sigma} \max_{x \in M} |R(x)||du|^4.$$

(7.4.32)

The existence of a-priori coercive estimates of this kind is one of the reasons why one studies the Neumann boundary condition as the natural boundary condition for the minimal surface with a boundary or for the harmonic maps with a boundary.
It turns out that, when the submanifold $Y \subset (M, \omega, J)$ is a Lagrangian submanifold, this Neumann boundary condition for a $J$-holomorphic map is equivalent to the Dirichlet boundary condition or the free boundary condition
\[ u(\partial \Sigma) \subset L \]
for $J$-holomorphic maps.

**Lemma 7.4.4** Let $g = \omega(\cdot, J \cdot)$ and denote by $NL = N^J L$ the normal bundle to $L$ with respect to $g$. Suppose $u$ is $J$-holomorphic and $u(\partial \Sigma) \subset L$. Then $u$ satisfies
\[ \frac{\partial u}{\partial \nu} \in N^J L. \]

**Proof** Let $S \subset \partial \Sigma$ be a connected component and let $(r, \theta)$ be cylindrical coordinates near $S$ so that the Hermitian metric $h$ has the form
\[ h = \lambda (dr^2 + d\theta^2). \]
Then $u$ satisfies
\[ \frac{\partial u}{\partial r} + J \frac{\partial u}{\partial \theta} = 0 \quad \text{on } S \quad (7.4.33) \]
and
\[ \frac{\partial u}{\partial \theta} = du \left( \frac{\partial}{\partial \theta} \right) \subset TL. \]
On the other hand, the Lagrangian property of $L$ and the compatibility of $J$ with $\omega$ imply that
\[ JTL = NL. \]
Therefore (7.4.33) implies that the normal derivative $\partial u/\partial r$ lies in $NL$. This finishes the proof. \( \square \)

By incorporating these results into the calculations done in the previous section, specialized for $J$-holomorphic maps $u$ satisfying a Lagrangian boundary condition, we have derived the following a-priori integral bound.

**Proposition 7.4.5** Let $L \subset (M, \omega, J)$ be Lagrangian. Suppose that $\Sigma$ is a compact surface with boundary $\partial \Sigma$. Let $u : \Sigma \rightarrow M$ be $J$-holomorphic and satisfy $u(\partial \Sigma) \subset L$. Then we have
\[ \int_{\Sigma} |\nabla du|^2 \leq \int_{\partial \Sigma} \max_{y \in Y} |B^y||du|^3 + \int_{\Sigma} \max_{z \in \Sigma} (\max(-K)|du|^2 + \frac{1}{2} \int_{\Sigma} \max_{x \in M} |R||du|^4. \]
This a-priori $W^{2,2}$ estimate enables us to derive the following a-priori estimate for the weak solutions of the $J$-holomorphic curve equation $\bar{\partial} J u = 0$. We refer the reader to Definition 8.5.1 for the definition of weak solutions.

**Corollary 7.4.6** Let $u$ be a weak solution for $\bar{\partial} J u = 0$. If $du \in L^4$ and $du|_{\partial \Sigma} \in L^3$, then $du \in W^{1,2}$, i.e., $u \in W^{2,2}$. 
Local study of $J$-holomorphic curves

In this chapter, we will study various local aspects of a $J$-holomorphic curve $u : \Sigma \to M$ both in the interior and on the boundary of the domain $\Sigma$. Starting from the calculations carried out in the last chapter, we first derive the main a-priori interior estimates following the standard practice of geometric analysis in the study of harmonic maps and the minimal surface theory. Then, following the more global approach used in (SU81), (Fl88a) and (Oh92), we derive the corresponding boundary estimates. After that we prove the fundamental ingredient, the $\epsilon$-regularity theorem following the argument used in (Fl88b), (Oh92). This regularity theorem is the crucial ingredient to achieve the derivative bound for a small-energy $J$-holomorphic map. Our proof of this $\epsilon$-regularity result uses the rescaling argument and is of a somewhat different flavor from the standard argument used in geometric analysis, e.g., it is different from that of (SU81), which relies on some geometric calculations. The same kind of rescaling argument is the cornerstone of Sacks and Uhlenbeck’s bubbling argument developed in (SU81), though.

After that, we derive isoperimetric inequality and the monotonicity formula and finally give the proof of the removal singularity theorem of finite-energy $J$-holomorphic curves with isolated singularities.

### 8.1 Interior a-priori estimates

In this section, we assume that $u$ is $J$-holomorphic. We fix any compatible metric $g$ and denote by $\nabla$ the canonical connection of the almost-Kähler manifold $(M, \omega, J)$ as given in Theorem 7.1.5. Then $u$ satisfies the equation (7.3.28).

We start by rewriting (7.3.28) into

$$|\nabla du|^2 = -\frac{1}{2} \Delta e(u) - 2\langle K \partial u, \partial u \rangle - 2\langle R(du, du)\partial u, \partial u \rangle. \quad (8.1.1)$$
The following standard inequality appearing in the theory of harmonic maps and minimal surfaces immediately follows from this identity.

**Lemma 8.1.1**  For any J-holomorphic map, we have the pointwise inequality

\[ \Delta e(u) \leq C_1 e(u) + C_2 e^2(u) \]  

(8.1.2)

for some universal constants \( C_1, C_2 \) depending only on \( \omega, J \). In fact, we can choose

\[ C_1 = 4 \max_{z \in \Sigma} |K(z)|, \quad C_2 = 4 \max_{x \in M} |R(x)|. \]  

(8.1.3)

We like to note that the constant \( C_1 \) in this lemma depends only on \( \Sigma \), and \( R \) depends only on \( M \) via \( \nabla \) (and hence on \( \omega, J \)).

**Remark 8.1.2**  At this point, we recall that the only requirement for \( h \) in the above calculations is that \( h \) is Hermitian with respect to the complex structure \( j \). Therefore we can vary the Hermitian metric \( h \) by multiplying \( \lambda^{-2} \) and consider \( \lambda^{-2}h \) instead. Then the new energy density function, denoted by \( e_\lambda \), has the form

\[ e_\lambda(u) = \lambda^2 e(u). \]

Since the area element \( dA_\lambda \) for the metric \( \lambda^{-2}h \) becomes \( \lambda^{-2}dA \) with \( dA \) being the area element of \( h \), the total energy

\[ E(u) := \frac{1}{2} \int_{\Sigma} |du|^2 dA \]

is preserved, i.e., \( E(u) = E_\lambda(u) \).

A natural geometric context in which such a family of metrics arises is when we pull back the restriction of the given metric \( h \) on the geodesic balls \( D_r^2(z_0) \subset \Sigma \) to the unit disc \( D(1) \subset \mathbb{C} \cong T_{z_0} \Sigma \) via the exponential map \( R_r: D^2(1) \to \Sigma \) given by \( R_r(v) = \exp_{z_0}(rv) \). The pull-back metrics \( R_r^*h \) on \( D^2(1) \) can be expressed as \( R_r^*h = \lambda_r(z)|dz|^2 \) for some functions \( \lambda_r \) on \( D^2(1) \) such that \( \lambda_r \to 1 \) in \( C^\infty \) as \( r \to 0 \). We also remark that if \( \lambda > 0 \) is a constant the Gauss curvature of \( \lambda^{-2}h \) is given by \( \lambda^2 K \) and so the constant \( C_{1,\lambda} \) for the metric \( \lambda^{-2}h \) can be taken such that

\[ C_{1,\lambda} = \lambda^2 C_1. \]  

(8.1.4)

In particular, by applying (8.1.2) to the metrics \( R_r^*h \) on \( D^2(1) \), we can make the constant \( C_{1,\lambda} \) as small as we want by letting \( r \to 0 \).
The following density estimate is crucial for the study of (interior) regularity of \( J \)-holomorphic maps. The main idea of the proof below is taken from that of Schoen (Sc84) for the study of harmonic maps.

Denote by \( D(r) \subset \Sigma \) a disc of radius \( r \) and by \( D(2r) \subset \text{Int} \Sigma \) the concentric disc of radius \( 2r \) smaller than the injectivity radius of \( h \) on \( \Sigma \).

**Theorem 8.1.3** (Interior density estimate) There exist constants \( C, \epsilon_0 \) and \( r_0 > 0 \), depending only on \( J \) and the Hermitian metric \( h \) on \( \Sigma \), such that, for any \( C^1 \) \( J \)-holomorphic map with \( E(r_0) \leq \epsilon_0 \), where

\[
E(r_0) := \frac{1}{2} \int_{D(r_0)} |du|^2,
\]

and discs \( D(2r) \subset \text{Int} \Sigma \) with \( 0 < 2r \leq r_0 \), \( u \) satisfies

\[
\max_{\sigma \in (0,r]} \left( \sigma^2 \sup_{D(r-\sigma)} e(u) \right) \leq CE(r) \tag{8.1.5}
\]

for all \( 0 < r \leq r_0 \). In particular, letting \( \sigma = r/2 \), we obtain

\[
\sup_{D(r/2)} |du|^2 \leq \frac{4CE(r)}{r^2} \tag{8.1.6}
\]

for all \( r \leq r_0 \).

**Proof** Let \( r_0 > 0 \) and consider any \( r \) such that \( 0 < r \leq r_0/2 \). Consider the function

\[
\sigma \mapsto \sigma^2 \sup_{D(r-\sigma)} e(u),
\]

which has values 0 at \( \sigma = 0 \) and \( r \). Therefore we can choose \( \sigma_0 \in (0, r) \) so that

\[
\sigma_0^2 \sup_{D(r-\sigma_0)} e(u) = \max_{\sigma \in (0,r]} \left( \sigma^2 \sup_{D(r-\sigma)} e(u) \right) > 0
\]

as long as \( u \) is not constant on \( D(r) \). (Note that, if \( u \) is constant on \( D(r) \) for some \( r > 0 \), there is nothing to prove.) Then we choose \( z_0 \in \overline{D(r-\sigma_0)} \) so that

\[
e(u)(z_0) = \sup_{D(r-\sigma_0)} e(u).
\]

By the choice of \( \sigma_0 \) and \( z_0 \), \( D_{\sigma_0/2}(z_0) \subset D(r-\sigma_0/2) \) and hence

\[
\sup_{D_{\sigma_0/2}(z_0)} e(u) \leq \sup_{D(r-\sigma_0/2)} e(u).
\]

On the other hand, we also have

\[
\left( \frac{\sigma_0}{2} \right)^2 \sup_{D(r-\sigma_0/2)} e(u) \leq \sigma_0^2 \sup_{D(r-\sigma_0)} e(u) \leq \sigma_0^2 e(u)(z_0).
\]
By combining the last two inequalities, we obtain

\[
\sup_{D_{\sigma_0/2}(z_0)} e(u) \leq 4e(u)(z_0). \tag{8.1.7}
\]

If \( \sigma_0^2 e(u)(z_0) \leq 4E(r) \), then (8.1.5) follows from the equality

\[
\sigma_0^2 e(u)(z_0) = \max_{\sigma \in [0,r]} (\sigma^2 \sup_{D_{r-\sigma}} e(u))
\]

by the definitions of \( \sigma_0, z_0 \).

So we may assume

\[
e(u)(z_0) > 4E(r)\sigma_0^{-2}. \tag{8.1.8}
\]

Now we apply a rescaling argument. Set

\[
e_0 = \frac{e(u)(z_0)}{E(r)}, \quad R_0 = \frac{e_0^{1/2}\sigma_0}{2}. \tag{8.1.9}
\]

Then it follows that \( e_0 \geq 4/\sigma_0^2 \) and \( R_0 \geq 1 \). Define a rescaled map \( g : D_{R_0}(0) \to M \) by

\[
g(y) = u\left(z_0 + y/e_0^{1/2}\right).
\]

Here we regard the rescaling map \( y \mapsto z_0 + y/e_0^{1/2} \) as the obvious rescaled exponential map \( \exp_{z_0}(y/e_0^{1/2}) \) from \( D_{R_0}(0) \subset T_{z_0}\Sigma \cong \mathbb{C} \) into \( D_{\sigma_0/2}(z_0) \subset \Sigma \). We compute

\[
e(g)(0) = |dg|^2(0) = |du(z_0)|^2/e_0 = E(r),
\]

where the last equality follows from (8.1.9).

Therefore it follows from (8.1.7) and (8.1.8) that \( g \) satisfies

\[
\sup_{D_{R_0}(0)} e(g) \leq 4E(r).
\]

Under the change of variables

\[
y \in D_{R_0}(0) \mapsto y + z_0/e_0^{1/2} \in D_{\sigma_0/2}(z_0),
\]

the metric \( h \) is pulled back to one on \( D_{R_0}(0) \). It follows from the remark around (8.1.4) that the maximum Gauss curvature \( K \) for the pull-back metric converges to zero as \( \sigma_0 \to 0 \).

We derive

\[
\sigma_0^2 \geq 4 \frac{E(r)}{e(u)(z_0)} = \frac{4}{e_0}
\]

and \( \max|K_{z_0}| \sim \lambda_0^2 \max|K| \) with \( \lambda_0 = 1/e_0 \). By choosing \( r_0 \) (and hence \( \sigma_0 \)) sufficiently small, we may assume that \( \max|K_{z_0}| \leq 2 \max|K| \) as long as \( \sigma_0 \) is sufficiently small.
From this and (8.1.2), we obtain
\[ \Delta e(g) \leq C_1 \sigma_0^2 + C_2 e(g) \leq \left( \frac{C_1 \sigma_0^2}{4} + C_2 4E(r) \right) e(g) \]
on $D_{R_0}(0)$. On the other hand, by applying the mean-value theorem of Morrey ((GT77) Theorem 9.20) on $D_1(0) \subset D_{R_0}(0)$, we obtain
\[ E(r) = e(g)(0) \leq C_0 \left( \frac{C_1 \sigma_0^2}{4} + C_2 4E(r) \right) \int_{D_1} e(g) \tag{8.1.10} \]
for some universal constant $C_0$ depending only on the coefficients of $\Delta$ in the geodesic coordinates. This constant can be chosen uniform for balls of fixed size, say $r \leq r_0$ with sufficiently small $r_0$, which can be chosen in a way depending only on the metric $h$. But we have
\[ \int_{D_1} e(g) \leq \int_{D_{R_0}} e(g) = \int_{D_{r_0/2(z_0)}} e(u) \leq \int_{D(r)} e(u) = 2E(r). \tag{8.1.11} \]
Combining (8.1.10) and (8.1.11), we obtain
\[ E(r) \leq 2C_0 \left( \frac{C_1 \sigma_0^2}{4} + C_2 4E(r) \right) E(r), \]
i.e.,
\[ 1 \leq 2C_0 \left( \frac{C_1 \sigma_0^2}{4} + C_2 4E(r) \right). \tag{8.1.12} \]
Recall that $\sigma \leq r \leq r_0$. First we choose $r_0 > 0$ to be
\[ r_0 = \sqrt{ \frac{1}{C_0 C_1} }. \tag{8.1.13} \]
By substituting this into (8.1.12) we obtain
\[ E(r) \geq \frac{1}{16C_0 C_2}. \tag{8.1.14} \]
Therefore, if we choose
\[ r_0 = \sqrt{ \frac{1}{C_0 C_1} }, \quad \epsilon_0 = \frac{1}{17C_0 C_2}, \]
the inequality (8.1.12) could not hold since we have $\sigma_0 \leq r_0$ and $E(r) \leq E(r_0) \leq \epsilon_0$. This finishes the proof of (8.1.5).

Then (8.1.6) follows from (8.1.5) for the choice of $r_0$, $\epsilon_0$ by taking $\sigma = r/2$ in (8.1.5), which finishes the proof. \(\square\)
8.2 Off-shell elliptic estimates

In the previous section, we mostly worked with the estimates ‘on shell’, i.e., with $J$-holomorphic maps. However, it is often useful to have a-priori elliptic estimates in the ‘off-shell’ level, i.e., ones for arbitrary smooth maps. This is especially so when one studies gluing problems of pseudoholomorphic curves. We will derive the off-shell estimates by refining the geometric calculations carried out in the previous sections applied to general smooth maps. We again exploit the Neumann boundary condition, rather than a totally real boundary condition, in our tensorial calculations involving the boundary conditions. As we pointed out before, a Lagrangian boundary condition implies the Neumann boundary condition for the $J$-holomorphic maps for any $\omega$-compatible $J$.

A different approach, which is more global, was used by the authors in (Fl88a), (Oh92) to obtain the local a-priori estimates of $J$-holomorphic curves and is based on the classical linear estimates on the Cauchy–Riemann equation. This approach applies also to totally real boundary conditions in the case with a boundary. Our new approach uses the precise tensorial calculations greatly facilitated by the usage of the canonical connection. This enables us to provide the precise geometric dependence of various coefficients that appear in the main $W^{2,4}$ estimates, Corollary 8.2.7.

We will always assume that our map is smooth, but need some completion of the space of smooth maps with respect to suitable Sobolev norms to perform an analytic study of $J$-holomorphic maps. We take the simplest route of defining such completions via the Nash embedding theorem (Nas56). In other words, we fix an isometric embedding $M \hookrightarrow \mathbb{R}^N$ for a sufficiently large $N \in \mathbb{N}$, and regard a map to $M$ as a vector-valued function whose values are contained in $M \subset \mathbb{R}^N$.

From now on, we consider a map $u : \Sigma \to M$ that satisfies the Neumann boundary condition

$$\frac{\partial u}{\partial v} \bigg|_{\partial \Sigma} \perp L \quad (8.2.15)$$

for a Lagrangian submanifold $L \subset (M, \omega)$.

**Definition 8.2.1** Let $k - 2/p > 0$, let $\| \cdot \|_{k,p}$ be the Sobolev $W^{k,p}$ norms of the vector-valued functions $u : \Sigma \to \mathbb{R}^N$ and denote by $W^{k,p}(D, \mathbb{R}^N)$ the space of $W^{k,p}$ functions. We define $W^{k,p}(\Sigma, \partial \Sigma, (M, R))$ to be the completion in $W^{k,p}(D, \mathbb{R}^N)$ of the set

$$C^\infty(\Sigma, \partial \Sigma, (M, R))$$

consisting of smooth maps $u$ satisfying $u(\partial \Sigma) \subset R$. 


8.2 Off-shell elliptic estimates

We denote by \( \| \cdot \|_{k,p,K} \) the \( \| \cdot \|_{k,p} \) norm restricted to the subset \( K \) of \( \Sigma \). Since we assume that \( k - 2/p > 0 \), all such maps are continuous and hence the boundary condition makes sense. Note that maps with finite \( W^{1,p} \) norm with \( p > 2 \) are continuous, so it makes sense for the maps in \( W^{1,p}(D, \partial D), (M, R) \) to satisfy \( u(\partial D) \subset R \).

We start with the identity

\[
\delta \nabla (du) = u^* T
\]
given in Lemma 7.3.2. We compute \( \Delta(du) \) using the definition

\[
\Delta(du) = d^\Sigma (d^\Sigma)^*(du) + (d^\Sigma)^* d^\Sigma (du).
\]

As usual, we also denote \( \delta^\Sigma = (d^\Sigma)^* \). Then, by the formula \( \delta^\Sigma = -* d^\Sigma * \) in a two-dimensional surface \( \Sigma \), the second term becomes

\[
(d^\Sigma)^* d^\Sigma (du) = -* d^\Sigma * (u^* T). \tag{8.2.16}
\]

Similarly to before, applying the Weitzenböck formula, we compute

\[
\Delta e(u) = \Delta \langle du, du \rangle = \langle \nabla^\Sigma \nabla du, du \rangle - 2|\nabla du|^2 + \langle du, \nabla^\Sigma \nabla du \rangle = 2\langle \Delta du, du \rangle + \langle -2K du, du \rangle - 2\langle R^\Sigma (du, du) du, du \rangle - 2|\nabla du|^2. \tag{8.2.17}
\]

The only difference of this formula from (7.3.27) is the appearance of the term \( 2\langle \Delta du, du \rangle \) on the right-hand side.

We compute

\[
\langle \Delta du, du \rangle = \langle d^\Sigma (d^\Sigma)^*(du) + (d^\Sigma)^* d^\Sigma (du), du \rangle = \langle d^\Sigma (d^\Sigma)^*(du), du \rangle - \langle * d^\Sigma * (u^* T), du \rangle. \tag{8.2.18}
\]

We now compute each term of the right-hand side. For this purpose, the following identity turns out to be very useful, as in (OhW12), from which we borrow its proof.

**Lemma 8.2.2** Assume that \( \alpha \) is a zero form in \( \Omega^0(u^* \xi) \) and \( \beta \) is a 1-form in \( \Omega^1(u^* \xi) \). \( \langle \cdot, \cdot \rangle \) is the inner product on \( u^* \xi \) introduced from the metric of \( Q \). Then we have

\[
\langle d^\Sigma \alpha, \beta \rangle - \langle \alpha, \delta^\Sigma \beta \rangle = -\delta \langle \alpha, \beta \rangle.
\]

**Proof** We compute

\[
-\delta \langle \alpha, \beta \rangle = *d * \langle \alpha, \beta \rangle = *d \langle \alpha, *\beta \rangle = *\langle d^\Sigma \alpha \wedge *\beta \rangle + *\langle \alpha, d^\Sigma (*\beta) \rangle
\]
= *(d^\nabla \alpha, \beta) \ d \text{vol} + \langle \alpha, *d^\nabla (*\beta) \rangle
= \langle d^\nabla \alpha, \beta \rangle - \langle \alpha, \delta^\nabla \beta \rangle.

In the third line, we also use the fact that our connection is a Riemannian connection and here one should extend the operation \wedge to the vector-forms in such a way that the product is taking the inner product in the fiber direction and take the wedge product on the base. □

Then we compute \langle *d^\nabla * (u^* T), du \rangle of \eqref{eq:B2.18}.

**Lemma 8.2.3**

\[ | - \langle *d^\nabla * (u^* T), du \rangle | \leq |T||\nabla du||du|^2 + |\nabla T||du|^3 \leq \frac{\epsilon}{2} |\nabla du|^2 + \frac{1}{2\epsilon} |T|^2 |du|^4 + |\nabla T||du|^3 \]

for all \(0 < \epsilon < 1\).

**Proof** Using Lemma 8.2.2 again, we compute

\[ -\langle *d^\nabla * (u^* T), du \rangle = -\langle d^\nabla * (u^* T), *du \rangle. \]

We then obtain the inequality by a straightforward calculation. □

For the following computation in this section, we will use the decomposition

\[ du = \delta_J u + \overline{\delta}_J u \in \Lambda^{(1,0)}(TM) \oplus \Lambda^{(0,1)}(TM). \]

Since we will not use the complexification and \(\delta_J u\) or \(\overline{\delta}_J u\) appearing in the decomposition \(du = \delta_J u + \overline{\delta}_J u \in \Lambda^{(1,0)}(T_C M) \oplus \Lambda^{(1,0)}(T_C M)\) in the previous chapter do not appear here, we will drop \(J\) and just denote \(\delta_J u = \delta u\) and \(\overline{\delta}_J u = \overline{\delta} u\) for the following computations in this section, which we hope does not confuse the reader.

For the term \langle d^\nabla (d^\nabla)^* (du), du \rangle of \eqref{eq:B2.18}, we have the following lemma.

**Lemma 8.2.4**

\[ \langle d^\nabla (d^\nabla)^* (du), du \rangle = 4|d^\nabla (\overline{\delta} u)|^2 + |u^* T|^2 - 4\langle u^* T, d^\nabla \overline{\delta} u \rangle - \delta (\delta^\nabla (du), du). \]

**Proof** Using \(\delta^\nabla = -*d^\nabla *\) and Lemma 8.2.2, we derive

\[ \langle d^\nabla (d^\nabla)^* (du), du \rangle = \langle d^\nabla \delta^\nabla (du), du \rangle \]

\[ = -\delta (\delta^\nabla (du), du) + \langle \delta^\nabla (du), \delta^\nabla (du) \rangle \]

\[ = -\delta (\delta^\nabla (du), du) + |\delta^\nabla (du)|^2. \]

Next we use the following identities:

\[ \delta^\nabla \partial u = J * d^\nabla \partial u, \]
\[ \delta^\nabla \overline{\partial} u = -J * d^\nabla \overline{\partial} u. \]
For the first, we compute
\[
\delta^\nabla \partial u = -*d^\nabla *\partial u = *d^\nabla \partial u \circ j = *d^\nabla J \partial u = J * d^\nabla \partial u.
\]

A similar computation leads to the second identity.

In particular, we obtain
\[
|d^\nabla \partial u| = |\delta^\nabla \partial u|, |d^\nabla \partial u| = |\delta^\nabla \partial u|.
\]

(8.2.19)

We derive
\[
|\delta^\nabla (du)|^2 = |-*d^\nabla *du|^2 = |d^\nabla *du|^2
= |d^\nabla * (\partial u + \bar{\partial} u)|^2 = |d^\nabla (\partial u j + \bar{\partial} u j)|^2
= |d^\nabla (J \partial u - J \bar{\partial} u)|^2 = |d^\nabla (\partial u - \bar{\partial} u)|^2
= |d^\nabla (du - 2 \bar{\partial} u)|^2
= |u^* T - 2 d^\nabla \bar{\partial} u|^2 = |u^* T|^2 + 4|d^\nabla \bar{\partial} u|^2 - 4\langle u^* T, d^\nabla \bar{\partial} u \rangle.
\]

Here we rewrite \( \bar{\partial} u = du - \bar{\partial} u \) and use the identity \( d^\nabla (du) = u^* T \) for the third equality. On substituting this into the identity at the beginning of this proof, we have finished the proof. \( \square \)

By substituting the formulae in Lemmata 8.2.3 and 8.2.4 into (8.2.18) and (8.2.17), and then re-arranging the terms, we get
\[
|\nabla du|^2 \leq -\frac{1}{2} \Delta e(u) - \delta\langle \delta^\nabla (du), du \rangle + \frac{\epsilon}{2} |\nabla du|^2 + \frac{1}{2\epsilon} |T|^2 |du|^4 + |\nabla T| |du|^3
- \langle Kdu, du \rangle - \langle R^\nabla (du, du)du, du \rangle
+ 4|d^\nabla (\partial u)|^2 + |u^* T|^2 - 4\langle u^* T, d^\nabla \bar{\partial} u \rangle.
\]

Therefore we obtain
\[
\left(1 - \frac{\epsilon}{2}\right) |\nabla du|^2 \leq -\frac{1}{2} \Delta e(u) - \delta\langle \delta^\nabla (du), du \rangle + \frac{1}{2\epsilon} |T|^2 |du|^4 + |\nabla T| |du|^3
- \langle Kdu, du \rangle - \langle R^\nabla (du, du)du, du \rangle
+ 4|d^\nabla (\partial u)|^2 + |u^* T|^2 - 4\langle u^* T, d^\nabla \bar{\partial} u \rangle.
\]

Taking the integral of this, we derive
\[
\left(1 - \frac{\epsilon}{2}\right) \int_\Sigma |\nabla du|^2 \leq \frac{1}{2} \int_\Sigma \frac{\partial}{\partial \nu} e(u) + \int_\Sigma \delta\langle \delta^\nabla (du), du \rangle + \frac{1}{2\epsilon} \int_\Sigma |T|^2 |du|^4
+ \int_\Sigma |\nabla T| |du|^3 - \int_\Sigma \langle Kdu, du \rangle - \langle R^\nabla (du, du)du, du \rangle
+ 4 \int_\Sigma |d^\nabla (\partial u)|^2 + \int_\Sigma |u^* T|^2 - 4 \int_\Sigma \langle u^* T, d^\nabla \bar{\partial} u \rangle.
\]

Here we remind the reader that our Laplacian \( \Delta \) acting on zero-forms (or functions) on \( \Sigma \) is the geometric Laplacian which is the negative of the classical Laplacian, which makes Green’s formula become
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\[-\frac{1}{2} \int_\Sigma \Delta e(u) \, dA = \frac{1}{2} \int_{\partial \Sigma} \frac{\partial}{\partial \nu} e(u) \, dt.\]

On \(\partial \Sigma\), we derive

\[
\frac{1}{2} \frac{\partial}{\partial \nu} e(u) \, dt + \langle \delta^\nabla (du), du \rangle - \langle *u^* T, *du \rangle
\]

\[
= \frac{1}{2} \left\langle \frac{\partial}{\partial r} \left| \frac{\partial u}{\partial t} \right|^2 + \frac{\partial u^2}{\partial r} \right\rangle \, dt - \left\langle \nabla_t \left( \frac{\partial u}{\partial r} \right), \frac{\partial u}{\partial r} \right\rangle \, dt - \langle *u^* T, *du \rangle
\]

\[
= \left\langle -2 \left( \nabla_t \frac{\partial u}{\partial t}, \frac{\partial u}{\partial r} \right) + \left( T \left( \frac{\partial u}{\partial r}, \frac{\partial u}{\partial t} \right) \right) dt - \langle *u^* T, *du \rangle \right\rangle \quad \text{(8.2.20)}
\]

where, for the penultimate equality, we used the identity

\[
\frac{1}{2} \left\langle \frac{\partial}{\partial r} \left| \frac{\partial u}{\partial t} \right|^2 \right\rangle = \left\langle \nabla_r \frac{\partial u}{\partial r}, \frac{\partial u}{\partial r} \right\rangle.
\]

We next compute

\[
- \langle *u^* T, *du \rangle = - \left( T \left( \frac{\partial u}{\partial r}, \frac{\partial u}{\partial t} \right), \frac{\partial u}{\partial t} \right) \, dt - \left( T \left( \frac{\partial u}{\partial r}, \frac{\partial u}{\partial t} \right), \frac{\partial u}{\partial r} \right) \, dt
\]

By substituting this into (8.2.20) and restricting our consideration to \(\partial \Sigma\), we obtain

\[
\frac{1}{2} \frac{\partial}{\partial \nu} e(u) \, dt + \langle \delta^\nabla (du), du \rangle - \langle *u^* T, *du \rangle
\]

\[
= -2 \left\langle \nabla_t \frac{\partial u}{\partial t}, \frac{\partial u}{\partial r} \right\rangle \, dt + \left( T \left( \frac{\partial u}{\partial r}, \frac{\partial u}{\partial t} \right), \frac{\partial u}{\partial r} \right) \, dt.
\]

At this point, we recall the definition of the second fundamental form of the canonical connection from Definition 7.4.1. With this definition, we can rewrite the last line of the last formula above as

\[
2 \left( B^\nabla \left( \frac{\partial u}{\partial t}, \frac{\partial u}{\partial r} \right), \frac{\partial u}{\partial r} \right) \, dt + \left( T \left( \frac{\partial u}{\partial r}, \frac{\partial u}{\partial t} \right), \frac{\partial u}{\partial r} \right) \, dt.
\]

Combining the above, we have obtained

**Theorem 8.2.5** Let \(\nabla\) be the canonical connection of the almost-Kähler manifold and \(R\) be any submanifold. Suppose \(u : (\Sigma, \partial \Sigma) \to (M, R)\) satisfy the Neumann boundary condition. Then
\[
\left(1 - \frac{\epsilon}{2}\right) \int_{\Sigma} |\nabla du|^2 \leq 4 \int_{\Sigma} |d^V(\tilde{\partial}u)|^2 + 2 \int_{\Sigma} |u^* T|^2 - 4\langle u^* T, d^V \tilde{\partial}u \rangle \\
- \int_{\Sigma} \langle K du, du \rangle + \langle R^V(du, du) du, du \rangle + \frac{1}{2\epsilon} \int_{\Sigma} |T|^2 |du|^4 \\
+ \int_{\Sigma} |\nabla T| |du|^3 + \int_{\partial \Sigma} 2 \left\langle B^V \left( \frac{\partial u}{\partial t}, \frac{\partial u}{\partial r} \right), \frac{\partial u}{\partial t} \right\rangle \\
+ \left\langle T \left( \frac{\partial u}{\partial r}, \frac{\partial u}{\partial t} \right), \frac{\partial u}{\partial t} \right\rangle dt
\] 
(8.2.21)
for any \(0 < \epsilon < 2\).

An immediate consequence with \(\epsilon = 1\) is the following main off-shell estimates.

**Theorem 8.2.6** Under the same hypothesis as Theorem 8.2.5
\[
\frac{1}{2} \int_{\Sigma} |\nabla du|^2 \leq 6 \int_{\Sigma} |d^V(\tilde{\partial}u)|^2 + 4 \int_{\Sigma} \max|T|^2 |du|^4 + \int_{\Sigma} \max(-K) |du|^2 \\
+ \frac{1}{2} \int_{\Sigma} |T|^2 |du|^4 + \int_{\Sigma} |\nabla T| |du|^3 + \int_{\Sigma} \max|R^V| |du|^4 \\
+ 2 \int_{\partial \Sigma} \max|B^V| |du|^3 + \max|T| |du|^3 dt.
\] 
(8.2.22)

An immediate corollary is the following off-shell counterpart of Proposition 7.4.5 with precise control of the coefficients in terms of the geometry of \((M, \omega, J)\) and the boundary manifold \(R^V\) and \(|\nabla(\tilde{\partial}u)|\). Here we also use the inequality \(|d^V(\tilde{\partial}u)| \leq |\nabla(\tilde{\partial}u)|\).

**Corollary 8.2.7** Under the same hypothesis as Theorem 8.2.5
\[
||du||_{1,2}^2 \leq 12 ||d^V(\tilde{\partial}u)||_{1,2}^2 + C_1 ||du||_{2}^2 + C_2 ||du||_{3}^2 + C_3 ||du||_{4}^2 + C_4 ||du||_{\partial \Sigma}^2
\]
for any smooth map \(u : (\Sigma, \partial \Sigma) \to (M, R)\) satisfying the Neumann boundary condition, for the constants
\[
C_1 = 2 \max(-K), \\
C_2 = 8 \max|T|^2 + 2 \max|R^V| + \max|T|^2, \\
C_3 = 2 \max|\nabla T|, \\
C_4 = 2 \max|B^V| + 2 \max|T|_l|
\]

**Remark 8.2.8**
(1) Recall that the torsion tensor \(T\) measures the failure of integrability of the almost-complex structure, the curvature \(R^V\) depends on the canonical connection \(\nabla\) and the second fundamental form \(B^V\) of \(R\) measures the
extrinsic curvature of the embedding \( R \subset M \). Therefore, if \((M, \omega, J)\) and \( R \) has bounded geometry, the coefficients \( C_i \) can be uniformly bounded.

(2) We would like to mention that the above estimates do not say anything about the \( C^0 \)-behavior of the map \( u : \Sigma \to M \) when \( M \) is non-compact and \( \Sigma \) is an open Riemann surface, or a punctured Riemann surface equipped with a metric that is cylindrical near the punctures. So the \( C^0 \) estimates become an issue that should be handled separately in that case.

Often the localized version of the above global estimate is also important, so an explanation is now in order. We denote by \( D \) either the open unit disc or the open semi-disc with boundary \((-1, 1)\). Denote by \((x, y)\) the standard coordinates \((x, y)\) and equip \( D \) with the standard flat metric. When \( D \) is a semi-disc, we impose the condition that \( u : D \to M \) satisfies the Neumann boundary condition. See Figure 8.1.

**Theorem 8.2.9** Let \( u : (D, \partial D) \to (M, R) \) be a smooth map with Neumann boundary condition with \( \|du\|_{1,4,D} =: C < \infty \). Then, for any given compact subset \( K \subset D \), we have

\[
\|du\|_{2,2,K}^2 \leq 6\|\overline{du}\|_{1,2,D}^2 + C_1\|du\|_{2,2,D}^2 + C_2\|du\|_{4,D}^4 + C_3\|du\|_{5,D\Sigma \cap D}^3,
\]

where the constants \( C_i \) depend continuously on \( J, K \) and \( \|du\|_{1,4,D} \).

**Proof** Choose a compactly supported smooth function \( \chi : \mathbb{R}^2 \to \mathbb{R} \) such that for another open disc (or semi-open disc) \( K \subset D' \subset D \) and

\[
\chi = \begin{cases} 
1, & \text{on } K, \\
0, & \text{outside } D'.
\end{cases}
\]

Then we estimate \( \|\nabla(\chi \cdot du)\| \) instead of \( \|\nabla(du)\| \). Noting that \( \|\nabla(du)\|_{1,4,K} \leq \|\nabla(\chi \cdot du)\|_{1,4,D} \) and splitting

\[
|\nabla(\chi du)| = |\nabla\chi du + \chi \nabla du|,
\]
it is straightforward to derive the required inequality. Now the coefficients \( C'_i \)
will also depend on \( \chi \) and \( \chi' \). We omit the detail of the derivation, leaving it as an exercise.

**Exercise 8.2.10** Complete the proof of the above theorem by providing the details which have been left out.

**Remark 8.2.11** The same kind of argument yields an estimate for \( \|du\|_{l-1,q,K} \)
for \( l \geq 3 \) with the constant \( C_1 \) depending on \( \|\nabla^{l-1}(J)\|_{\infty} \) and \( \|du\|_p \); however, that will not be used in this book.

### 8.3 Removing boundary contributions

So far, we have emphasized the usage of canonical connection in our computations. However, we will need to have the on-shell integral formula for the Neumann boundary conditions to derive some inequality of the type

\[
\|du\|_{\infty, D(r)}^2 \leq CE(u|_{D(2r)}) = C\|du\|_{L,D(2r)}^2
\]

in order to derive the boundary analog of the density estimate (8.1.3) later in the proof of the removal of boundary singularities. For this purpose, it turns out that it is important to get rid of the boundary contributions from (7.4.31). This can be done if one uses the Levi-Civita connection of a metric with respect to which the boundary manifold \( L \) becomes totally geodesic. We explain in this section how one can make this adjustment in the off-shell level.

For computational purposes, the following lemma, whose proof we borrow from p. 683 of (Ye94), is useful.

**Lemma 8.3.1** There exists another Hermitian metric \( g \) on \( M \) such that \( N \) satisfies

\[
JT_pN \perp T_pN
\]

for every \( p \in N \).

**Proof** Let \( \{e_1, \ldots, e_n\} \) be a local orthonormal frame on \( N \). Since \( N \) is totally real with respect to \( J \), \( \{e_1, \ldots, e_n, Je_1, \ldots, Je_n\} \) is a local frame of \( TM|_N \) on \( N \).

By a partition of unity we obtain a metric \( \tilde{g} \) on the vector bundle \( TM|_N \), which we extend to a tubular neighborhood of \( N \). Then we set \( \tilde{g} = \frac{1}{g(J,J)} + \tilde{g} \). By interpolating between \( g \) and \( \tilde{g} \), we obtain a Hermitian metric \( g_0 \).
Remark 8.3.2 We note that any metric on a vector bundle $E \to N$ can be extended to a metric on $E$ as a manifold so that the latter becomes reflection-invariant on $E$. In particular, the zero section of $E$ becomes totally geodesic. Using this, it is easy to improve the above lemma so that $N$ becomes totally geodesic in addition with respect to the resulting metric $g$ in the lemma.

We then note that

$$\nabla - \nabla^{LC} = P$$

for some $(2, 1)$-tensor $P$. Therefore we derive

$$\nabla_X(du) = \nabla^{LC}_X(du) + P(X, du)$$

and hence

$$|\nabla(du)|^2 \leq 2(|\nabla^{LC}(du)|^2 + |P(\cdot, du)|^2). \quad (8.3.25)$$

By the totally geodesic property of the boundary $L$, one can establish a variation of the integral identity of the type $(7.4.31)$ without boundary contributions. All other interior integrals will be bounded by $|du|^2$, $|du|^3$ and $|du|^4$ with coefficients involving $|\nabla^{LC} J|$ and $|\nabla^{LC} J^2|$ whose norm is bounded by the constants independent of the map $u$ but depend only on $J$ and $g$. We also recall that, for the associated Levi-Civita connection, we have $T \equiv 0$ and $B \equiv 0$ on $L$. This gives rise to the integral inequality

$$\int_\Sigma |\nabla^{LC}(du)|^2 \leq C \int_\Sigma (|\nabla(\bar{\partial}u)|^2 + |du|^2 + |du|^3 + |du|^4)$$

for some constant $C > 0$. We would like to emphasize that this estimate does not involve the boundary integrals at all.

Exercise 8.3.3 Verify the above claim by performing the relevant tensor calculations analogous to the one carried out for the proof of Theorem 8.2.6.

On combining this with $(8.3.25)$, we have obtained the following integral inequality.

Proposition 8.3.4 Let $\nabla$ be the canonical connection and let $u : (\Sigma, \partial \Sigma) \to (M, L)$ be a smooth map satisfying the Neumann boundary condition. Then there exists a constant

$$\int_\Sigma |\nabla(du)|^2 \leq C' \int_\Sigma (|\nabla(\bar{\partial}u)|^2 + |du|^2 + |du|^3 + |du|^4).$$

Furthermore, for any disc $D(r) \subset D(r') \subset \Sigma$ of radius $r > 0$, we have the inequality
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\[ \int_{D(r)} |\nabla(\bar{\partial}u)|^2 \leq C' \int_{D(r')} (|\nabla(\bar{\partial}u)|^2 + |du|^2 + |du|^3 + |du|^4) \]

for a constant \( C' = C'(r, r') \). In particular,

\[ \int_{D(r)} |\nabla(du)|^2 \leq C' \int_{D(r')} (|du|^2 + |du|^3 + |du|^4) \quad (8.3.26) \]

for any \( J \)-holomorphic map \( u \).

Here we refer to Theorem 8.2.9 in the previous section for the details of the derivation of the local estimates from the global estimates.

Once we have derived, we can replace 4 by any \( p > 2 \) by the Sobolev embedding and by the following standard interpolation inequality (see, e.g., (7.10) in \((GT77)\)):

\[ \|\xi\|_r \leq \delta \|\xi\|_s + \delta^{-a} \|\xi\|_q \text{ for any } \delta > 0 \]

where

\[ a = \frac{1/q - 1/r}{1/r - 1/s}. \]

**Theorem 8.3.5** (Main off-shell elliptic estimates) Let \( l > k \) and \( l - 2/q > k - 2/p \) with \( q, p > 2 \), and let \( J \) be any compatible almost-complex structure. For any given compact subset \( K \subset D \), there exists \( C = C(\|du\|_p, J, K) \) continuously depending on \( \|du\|_p, J \) such that, for all smooth \( u \) (and hence for all \( u \in W^{1,p}(D, M) \)), we have

\[ \|du\|_{1,q,K} \leq C(\|\bar{\partial}ju\|_{1,q} + \|du\|_p). \quad (8.3.27) \]

If \( J \) varies in a compact set of \( \mathcal{J}_\omega \), there exist uniform bounds \( C \) depending continuously on \( J \).

The following corollary will be useful for the perturbation argument to construct a genuine solution out of the given approximate solutions.

**Corollary 8.3.6** If \( u_\alpha \) is a smooth sequence such that \( \|du_\alpha\|_p \) is bounded and \( \lim_\alpha \|\bar{\partial}ju_\alpha\|_{1,p} = 0 \), then there exists a subsequence of \( u_\alpha \) converging in \( W^{2,q}(K, M) \) to some \( J \)-holomorphic map \( u : K \to M \).

The following is a curious question to ask.

**Question 8.3.7** Can we avoid using the above-mentioned special metric in the derivation of the estimate without boundary contributions of the type above but just keep using the compatible metric and the associated canonical connection?
8.4 Proof of \(\epsilon\)-regularity and density estimates

The proof of the main energy density estimate, Theorem 8.1.3, for the interior uses the pointwise differential inequality

\[
\Delta e(u) \leq C_1 e(u) + C_2 e(u)^2
\]

together with the mean-value theorem in an essential way. This local argument cannot be directly applied to the boundary estimate. Therefore we will follow the more global method used in (Oh92) to obtain the boundary estimate as well as the interior in a unified fashion. This strategy in turn largely adapts the global approach taken by Sachs and Uhlenbeck in their study of harmonic maps in (SU81) to the current pseudoholomorphic context. In particular, we apply their scheme to the boundary-value problem. The main ingredient of obtaining the a-priori estimates for a \(J\)-holomorphic map with small energy is the usage of symplectic geometry combined with the celebrated Sachs–Uhlenbeck rescaling argument in (SU81). This simplifies the study of the boundary estimate, which otherwise would rely on some non-trivial study of the free-boundary-value problem as carried out by Ye (Ye94).

We always assume that \(M\) is either closed or has a bounded geometry when it is non-compact. The main theorem that we would like to prove in this section is the following.

Denote by \(\hat{D}\) either the open unit disc or the open semi-disc with boundary \((-1, 1) \subset \mathbb{C}\). See Figure 8.1.

**Theorem 8.4.1**  Let \(R \subset M\) be a totally real submanifold. Suppose that \(u : D \to M\) with \(u(\partial D) \subset R\), and \(\bar{\partial}_J u = 0\). Then there exists \(\epsilon > 0\), such that, if \(\int_D |du|^2 < \epsilon\), we have \(\|du\|_{p,D'} \leq K_4\) for any \(p \geq 2\) and for any smaller disc \(D'\) with \(\overline{D'} \subset D\), where \(K_4\) depends on \(\epsilon\), \(p\), \(J\) and \(D'\).

This immediately gives rise to the following derivative bound, which is often called the \(\epsilon\)-regularity theorem in the literature of geometric measure theory.

**Corollary 8.4.2**  (The \(\epsilon\)-regularity theorem)  Let \(D' \subset D\) be a smaller disc with \(\overline{D'} \subset \hat{D}\) and suppose that \(u\) satisfies the same condition as in Theorem 8.4.1. Then \(\|du\|_{\infty,D'} \leq K_5\) for some constant \(K_5 > 0\).

**Proof**  We choose another disc \(D''\) with \(\overline{D'} \subset D''\) and \(\overline{D''} \subset \hat{D}\). We apply Theorem 8.4.1 to the pair \((\overline{D'}, D'')\) and obtain the inequality \(\|du\|_{1,p,\overline{D'}} \leq K_6\). Then the Sobolev inequality
\[ \|du\|_{\infty, D'} \leq C\|du\|_{1,p, \overline{D'}} \]

proves the corollary. \[\square\]

**Remark 8.4.3** Let \( u : \Sigma \to M \) be a smooth map and consider its restriction \( u|_{B_{\eta}(x)} : B_{\eta}(x) \to M \) to a disc \( B_{\eta}(x) \subset \Sigma \). Fix a flat metric on \( B_{\eta}(x) \) regarding \( B_{\eta}(x) \) as a subset of the unit disc \( D(1) \subset \mathbb{C} \). Consider the rescaled map \( u_{\eta} : D(1) \to M, u_{\eta}(z) = u(x + \eta z) \). Then we have

\[
\int_{D(1)} |du_{\eta}|^p = \eta^{p-2} \int_{B_{\eta}(x)} |du|^p \iff \|du_{\eta}\|_{p,D(1)} = \eta^{1-2/p}\|du\|_{p,B_{\eta}(x)}.
\]

Note that, when \( p = 2 \), this rescaling does not change the \( L^p \)-norm of derivatives, but that, for \( p > 2 \), \( \|du_{\eta}\|_{p,D(1)} \to 0 \) as \( \eta \to 0 \).

**Proof of Theorem 8.4.1** We will prove this by contradiction. Suppose to the contrary that there exists a disc \( D' \subset D = D(1) \) with \( \partial D' \subset \bar{D} \) and a sequence \( \{u_{\alpha}\} \) such that \( u_{\alpha}(\partial D) \subset \partial R \), where \( R \) is a totally real submanifold, and \( \bar{\partial}J_{u_{\alpha}} = 0 \) satisfying

\[
\int_{D(1)} |du_{\alpha}|^2 \to 0, \quad \|du_{\alpha}\|_{p,D'} \to \infty \quad (8.4.28)
\]
as \( \alpha \to \infty \). Define

\[
\eta_{\alpha}(D') = \inf\{\eta > 0 \mid \exists x \in \bar{D}' \subset \bar{D} \text{ such that } \eta^{1-2/p}\|du_{\alpha}\|_{p,B_{\eta}(x)} \geq 1\}.
\]

Clearly, for each fixed \( \alpha \), \( \eta_{\alpha}(D') > 0 \): otherwise it would imply that there exists a sequence \( \eta_j \to 0 \) and \( x_j \in \bar{D}' \) such that

\[
\eta_j^{1-2/p}\|du_{\alpha}\|_{p,B_{\eta_j}(x_j)} \geq 1
\]

and hence

\[
\|du_{\alpha}\|_{p,B_{\eta_j}(x_j)} \geq \eta_j^{-1+2/p} \to \infty.
\]

But this contradicts the hypothesis \( \|du_{\alpha}\|_{p,D} < \infty \) because

\[
\|du_{\alpha}\|_{p,D} \geq \|du_{\alpha}\|_{p,B_{\eta_j}(x_j)}
\]

for all \( j \).

Now let us vary \( \alpha \). We first note that, by the definition of \( \eta_{\alpha}(D') > 0 \), we obtain

\[
\eta_j^{2-p}\|du_{\alpha}\|^2_{B,B_{\eta_j}(x)} \leq 1
\]

for all \( x \in \bar{D}' \), whenever \( \eta < \eta_{\alpha}(D') \). If \( 0 < \eta_{\alpha}(D') \) is bounded away from zero for some subsequence \( u_{\alpha \prime} \), we can choose \( \eta_0 \) such that
We cover $D'$ by a finite number, say $N_{\eta_0}$ independent of $\alpha$, of closed balls $B_{\eta_0}(x_{\alpha,k}) \subset D$ for $k = 1, \ldots, \eta_0$ such that $x_{\alpha,k} \in \overline{D'}$ and $\eta_0^{1-2/p}\|du_\alpha\|_{p,B_{\eta_0}(x_{\alpha,k})} \leq 1$. Then

$$\|du_\alpha\|_{p,B_{\eta_0}(x)} \leq \eta_0^{2-p}. $$

By adding up this inequality over the closed balls and taking the $p$th square root thereof, we obtain the bound

$$\|du_\alpha\|_{p,D} \leq CN_{\eta_0}\eta_0^{2/p-1}$$

for all sufficiently large $\alpha$. This will contradict the standing hypothesis that $\|du_\alpha\|_{p,D} \to \infty$ as $\alpha \to \infty$.

Therefore we have proved that $\eta_\alpha(D') \to 0$ as $\alpha \to 0$. Using the definition of $\eta_\alpha(D')$, continuity of $\eta_\alpha^{2-p}\|du_\alpha\|_{p,B_\eta(x)}$ over $\eta$ and the above remark, we can choose $\eta_\alpha$ arbitrarily close to but smaller than $\eta_\alpha(D')$ for each given $\alpha$ so that

$$\int_{B_{\eta_\alpha}(x_{\alpha,k})} |du_\alpha|^p \geq \frac{1}{2}\eta_\alpha^{2-p}$$

for some $x_{\alpha} := x_{\alpha,k} \in \overline{D'}$ and

$$\eta_\alpha^{p-2}\|du_\alpha\|_{B,B_{\eta_0}(x)} \leq 1$$

for all $x \in \overline{D'}$.

Denote

$$r_\alpha = \eta_\alpha^{-1}\min\{\text{dist}(x_{\alpha}, \partial D), \text{im}(x_{\alpha})\}. $$

After passing to a subsequence, there are two cases to consider:

1. $r_\alpha \to \infty$,
2. $r_\alpha \to r < \infty$.

We will consider the two cases similarly and derive a contradiction to the standing hypothesis (8.4.28) by the rescaling argument used in (SU81).

Consider case (1) first. We define the rescaled map

$$v_\alpha(z) := u_\alpha(\eta_\alpha z + x_\alpha). $$

The domain of $v_\alpha(z)$ is

$$\{z \in \mathbb{C} \mid \eta_\alpha z + x_\alpha \in D\}. $$

However,

$$\begin{cases}
|\eta_\alpha z + x_\alpha| \leq 1 \\
\text{im}(\eta_\alpha z + x_\alpha) \geq 0
\end{cases} \iff \begin{cases}
|z + x_\alpha/\eta_\alpha| \leq 1/\eta_\alpha \to \infty \\
\text{im}(z) \geq -\text{im}(x_\alpha/\eta_\alpha).
\end{cases}$$
We note that \(1/\eta - |x_\alpha|/\eta = (1 - |x_\alpha|)/\eta \to \infty\) since \(1 - |x_\alpha| > \eta_0 > 0\) and \(\text{im}(x_\alpha/\eta) \to \infty\) as \(\alpha \to \infty\). Therefore, for any given \(R > 0\), the domain of \(v_\alpha(z)\) eventually contains \(B_R(0)\).

Furthermore, we may assume that
\[
B_R(0) \subset \left\{ z \in \mathbb{C} \mid \eta_\alpha z + x_\alpha \in \overline{D} \right\}.
\]

Therefore, the maps \(v_\alpha : B_R(0) \to M\) satisfy the following properties:

1. \(\|dv_\alpha\|_2 \to 0\) (from the scale invariance of the \(L^2\)-norm)
2. \(\|dv_\alpha\|_{p,B_R(0)} \geq \frac{1}{2}\) by (8.4.29)
3. \(\|dv_\alpha\|_{p,B_R(x)} \leq 1\) for all \(x \in B_{R^{-1}}(0)\)
4. \(\tilde{\partial} Jv_\alpha = 0\).

Here (3) follows from (8.4.30).

For each fixed \(R\), we take the limit of \(v_\alpha|_{B_R}\). Applying (3) and then the main local estimate, we obtain
\[
\|dv_\alpha\|_{p,B_{8/10}(x)} \leq C
\]
for some \(C = C(R)\). By the Sobolev embedding theorem, we have a subsequence that converges in the Hölder space \(C^\epsilon\) for some \(\epsilon > 0\) in each \(B_{8/10}(x), x \in \overline{D}'\). Then, by applying the main elliptic estimates, Theorem 8.3.5, and the bootstrap arguments we derive that the convergence is in \(C^\infty\) topology on \(B_{8/10}(x)\) and in turn on \(B(R)\).

Therefore the limit \(w_R : B_R(0) \to M\) satisfies the following statements.

1. \(\tilde{\partial} Jw_R = 0\).
2. \(\|dw_R\|_2^2 \leq \limsup \|dv_\alpha\|_{B_R(0)}^2 \to 0\) and hence \(\|dw_R\|_2^2 = 0\).
3. Since \(v_\alpha \to w_R\) converges in \(C^1\), we have
   \[
   \|dw_R\|_{p,B_1(0)}^2 = \lim_{\alpha \to \infty} \|dv_\alpha\|_{p,B_1(0)}^2 \geq \frac{1}{2}.
   \]
Then statements (2) and (3) clearly contradict each other and so case (1) cannot occur.

Next we consider case (2), i.e., let \(r_\alpha \to r < \infty\). Since we assume \(D' \subset \overline{D}' \subset D\), especially for the semi-disc case, the minimum \(r_\alpha\) is realized by \(\eta_\alpha^{-1} \text{im}(x_\alpha)\) eventually. In this case,
\[
v_\alpha(z) := u_\alpha(\eta_\alpha z + x_\alpha)
\]
is defined on
\[
\text{Dom} v_\alpha = \{ z \in \mathbb{C} \mid \eta_\alpha z + x_\alpha \in D \}.
\]
The convergence $x_\alpha \to \partial D$ is as fast as that of $\eta_\alpha \to 0$. (See Figure 8.2.)

More precisely, we have

$$\begin{cases}
|\eta_\alpha z + x_\alpha| < 1 \\
\Im(\eta_\alpha z + x_\alpha) \geq 0
\end{cases} \iff \begin{cases}
|z + x_\alpha/\eta_\alpha| < 1/\eta_\alpha \to \infty \\
\Im(z) \geq -\Im(x_\alpha/\eta_\alpha) \to -r.
\end{cases}$$

In this case, the domain of $v_\alpha$ will eventually contain

$$\{z \in C \mid |z| \leq R, \Im(z) \geq -r\}$$

for each given $R > 0$. Taking the same limit procedure, we will again get a contradiction, which shows that case (2) cannot occur either.

This proves the existence of $\epsilon > 0$ required in Theorem 8.4.1 and hence finishes the proof.

Next we prove the following uniform estimate.

**Proposition 8.4.4** Let $u$ and $\epsilon$ be as in the $\epsilon$-regularity theorem (i.e. $\int |\nabla u|^2 < \epsilon, \bar{\partial}_J u = 0$). Then, for any $0 < r < r' < 1$, $\max_{|z| \leq r'} |du(x)| \leq CE(u|_{D(r')})$, where $C = C(r, r')$ does not depend on $u$ but depends only on $r$ and $\epsilon$.

**Proof** Choose $r'$ so that $0 < r < r' < 1$. Apply the main elliptic estimate to $K = D(r)$ and $D = D(r')$. (Note that $D$ is not the unit disc here, but one can obtain a similar estimate by rescaling the map. Then constants in the estimate will change but will depend only on $r$.)

On the other hand, since $E(u)|_{D(r')} \leq E(u) \leq \epsilon$, we have $|du| \leq K_5$ for some constant $K_5$ that is independent of $u$ by the $\epsilon$-regularity theorem. Then we can convert the inequality in Proposition 8.3.4 for $u$ satisfying $\bar{\partial}_J u = 0$ into

$$\int_{D(r')} |\nabla (du)|^2 \leq C \int_{D(r')} |du|^2 (1 + |du| + |du|^2) \leq C \int_{D(r')} (1 + K_5 + K_5^2)|du|^2.$$
Setting $C' = C(1 + K_5 + K_5^2)$, we derive
\[ \|du\|_{1,2;D(r'')} \leq C \cdot \|du\|_{2,D(r')} \]
for $r < r'' < r'$. Then, by the Sobolev inequality,
\[ \max_{x \in D(r')} |du(x)| \leq K' \cdot \|du\|_{1,2,D(r'')} \leq K' K(r) \cdot \|du\|_{2,D(r')} \]
for some $K', K(r) > 0$. On setting $C = K' K(r)$ and noting that $\|du\|_{2,D(r')} = E|du|_{D(r')}$, we have finished the proof. \[ \Box \]

**Remark 8.4.5** The proof of the above proposition rectifies an error in the proof of the corresponding statement in (Oh92).

The following crucial estimate is the boundary analog of Theorem 8.1.3.

**Proposition 8.4.6** (Boundary energy density estimate) Let $U$ and $\epsilon$ be as above, but we assume that $U$ is defined on $D(2)$. Then, for $x \in D(\frac{1}{2})$, we have
\[ |x| |du(x)| \leq \sqrt{C \left( \int_{D(2|x|)} |du(x)|^2 \right)^{1/2}}, \]
or, equivalently,
\[ |du(x)|^2 \leq \frac{CE(2r)}{r^2}, \quad r = |x|. \]

**Remark 8.4.7** If we consider $v : D(1) \rightarrow M$ by $v(z) = u(|x|z)$, then
\[ |dv(0)| \leq C \int_{D(2)} |dv(z)|^2. \quad (8.4.31) \]

Note that this is the boundary analog to the mean-value inequality (8.1.10). We also remark that the proof of this proposition applies to the interior estimates given in Theorem 8.1.3.

**Proof** Let $x_0 \in D(\frac{1}{2})$ and consider $v(z) = u(x_0 + |x_0|z)$ on $N(\frac{3}{2}, x_0)$, where
\[ N(r, x_0) = \left\{ z \in D(r) \mid \Im z \geq \frac{\Im x_0}{x_0} \right\}. \]
If $x_0 \in D(\frac{1}{2})$, then we have conformal equivalence,
\[ (N(3/2, x_0), \partial N(3/2, x_0)) \sim (D(1), \partial D(1)), \]
where the derivatives of $\phi$ are uniformly bounded over the choice of $x_0 \in D(\frac{1}{2})$. 

\[ \sqrt{C \left( \int_{D(2|x|)} |du(x)|^2 \right)^{1/2}}, \]
\[ |du(x)|^2 \leq \frac{CE(2r)}{r^2}, \quad r = |x|. \]
Considering $v \circ \phi^{-1} : D(1) \to M$ and using the fact that the energy is conformally invariant, we have

$$|dv(0)| \leq C \cdot \|dv\|_{2,D(1)}.$$ 

However, $dv(0) = |x_0| \cdot du(x_0)$, and

$$\|dv\|_{2,D(1)} = \|du\|_{2,N(\frac{1}{2},x_0)} \leq \|du\|_{2,D(2|x_0|)}.$$

On combining these, we have $|x_0|\|du(x_0)\| \leq C \cdot \|du\|_{2,D(2|x_0|)}$. By varying $x_0$, we have finished the proof. □

In fact, we can quantify how large the $\epsilon$ appearing in the $\epsilon$-regularity theorem can be. For this purpose, we introduce the geometric quantity

$$A(\omega, L; J) = \min\{A_S(\omega, J), A_D(\omega, J; L)\} > 0,$$

where

$$A_S(\omega, J) := \inf\left\{ \int v^* \omega > 0 \mid v : S^2 \to M, \bar{\partial}_J v = 0 \right\}$$

and for a given Lagrangian submanifold $L \subset (M, \omega)$

$$A_D(\omega, L; J) := \inf\left\{ \int w^* \omega > 0 \mid w : (D^2, \partial D^2) \to (M, L), \bar{\partial}_J w = 0 \right\}.$$

**Proposition 8.4.8** Let $L$ be a Lagrangian submanifold of $M$. Then

1. $A(\omega, L; J) > 0$.
2. For any given $\delta > 0$, the statement of Theorem 8.4.1 uniformly holds for all $w$ satisfying $E_J(w) < A(\omega, L; J) - \delta$, where all the a-priori constants appearing in the estimates depend not on $w$ but only on $\delta$ (in addition to $L$ and $J$).

**Proof** We first prove the statement (1) by contradiction. We first make the following preparation before starting with actual proof of the proposition. Applying the Darboux–Weinstein theorem, we consider a diffeomorphism $\psi : V \to U$, where $V$ is a neighborhood of the zero section of $T^*L$ and $U$ is a neighborhood of $L$ in $M$ such that

$$\psi^* \omega = \omega_0, \ \omega_0 = -d\theta,$$

where $\theta$ is the canonical one-form on $T^*L$. Note that $\theta \equiv 0$ on the zero section of $T^*L$. Suppose that $A(\omega, L; J) = 0$. Then there exists a sequence of $J$-holomorphic \{u_\alpha\} such that

$$u_\alpha|_{\partial L} \subset L, \ E(u_\alpha) > 0, \ E(u_\alpha) \to 0, \ \bar{\partial}_J u_\alpha = 0.$$
From the uniform estimate, Proposition 8.4.4, on $|du_\alpha|$ in terms of energy, $|du_\alpha|$ converges uniformly to zero. In particular, for sufficiently large $\alpha$, the image of $u_\alpha$ will be contained in a Darboux neighborhood of $L$. Then we have

$$0 < \frac{1}{2} E(u_\alpha) = \int_{\Sigma} u_\alpha^* \omega = -\int_{\partial \Sigma} u_\alpha^* \theta = 0,$$

since $u_\alpha|_{\partial \Sigma} \subset L$ and $\theta \equiv 0$ on $L$. This is a contradiction and finishes the proof of Statement (1).

A careful examination of the above proof will also give rise to a contradiction as long as we assume that $E(u_\alpha) < A(\omega, L; J) - \delta$ for any given $\delta > 0$. We leave the details as an exercise. \hfill $\Box$

**Exercise 8.4.9**  Complete the proof of Statement (2) above.

This proposition motivates the following definition of an interesting symplectic invariant first exploited by Chekanov (Che98). See also (Oh97c) and (Oh05d) for the usages of such $\epsilon$-regularity-type invariants.

**Definition 8.4.10**  Assume that $(M, \omega)$ is tame and let $J_\omega$ be the set of tame almost-complex structures. For any compact Lagrangian submanifold $L$ without a boundary, we defined

$$A(L; M, \omega) := \sup_{J \in J_\omega} A(\omega, J; L).$$

Obviously by definition $A(L; M, \omega) = \infty$ if there exists a $J \in J_\omega$ for which there exists neither a $J$-holomorphic sphere nor a $J$-holomorphic disc with a boundary contained in $L$. For example, if $L$ is weakly exact, then $A(L; M, \omega) = \infty$.

**Corollary 8.4.11**  Suppose that $M$ is closed and $L$ is weakly exact. Consider a given bordered Riemann surface $(\Sigma, j)$ and $(j, J)$-holomorphic maps $u : (\Sigma, \partial \Sigma) \rightarrow (M, L)$ with finite energy. Let $E_{(j, J)}(u) < \infty$. Then there exists a constant $C > 0$ depending only on $E$ such that

$$|du|_C^0 \leq C$$

for all $J$-holomorphic maps $u$ with $E_{(j, J)}(u) \leq E$.

**Remark 8.4.12**  Note that Proposition 8.4.8 is trivial if $L \subset (M, \omega)$ is rational. On the other hand, when $L$ is irrational, there is no topological origin for the positivity of $A(\omega, L; J)$. In this case, the positivity comes from the geometric origin, or more precisely from the $\epsilon$-regularity result. In some literature, it is
stated that this corollary is a consequence of Gromov’s compactness theorem. This is a misleading statement because the proof of Gromov’s compactness theorem itself for the irrational \( L \subseteq (M, \omega) \) relies on this corollary.

Now we briefly study the case of the totally real boundary condition \( R \) of a general almost-complex manifold \((N, J)\). We closely follow the exposition given by Ye in (Ye94) for this, with slight modification of his second-order method to a first-order one.

We fix an almost-Hermitian metric \( g \) on \( N \) given in Lemma 8.3.1 and define the constants

\[
A(J, R; g) := \{ w : D^2 \rightarrow N \mid \bar{\partial}_J w = 0, \ w(\partial D^2) \subset R, \ w \text{ is non-constant} \},
\]

\[
A(J; g) := \{ v : S^2 \rightarrow N \mid \bar{\partial}_J v = 0, \ v \text{ is non-constant} \}.
\]

The following proposition was proved by Ye (Ye94) as a special case of the general harmonic-map-type equation with free boundary condition. Here we employ a purely first-order method involving a pseudoholomorphic curve equation with a Neumann boundary condition.

**Theorem 8.4.13** Let \((N, J, g)\) be an almost-Hermitian manifold with bounded geometry. Then \( A(J; g) > 0 \), and, if \( R \subseteq (N, J) \) be a compact totally real submanifold, \( A(J, R; g) > 0 \).

**Proof** We will consider only the case of \( A(J, R; g) \), leaving the case of \( A(J; g) \) to the reader. Suppose that \( A(J, R; g) = 0 \). Then there is a subsequence \( \{w_\alpha\} \) such that

\[
w_\alpha|_{\partial \Sigma} \subset R, \ E(w_\alpha) > 0, \ E(w_\alpha) \to 0 \text{ and } \bar{\partial}_J w_\alpha = 0.
\]

Since \( E(w_\alpha) \to 0, \ E(w_\alpha) < \epsilon \) eventually, where \( \epsilon \) is the one in the \( \epsilon \)-regularity estimate.

By covering \( \Sigma \) into a finite union of unit discs and semi-discs, so that the discs of radius \( \frac{1}{2} \) already cover \( \Sigma \), we have a uniform bound for all \( \| \cdot \|_{1,p} \)-norms.

In particular, \( w_\alpha \) converges to a constant map, say \( p \in R \), and so the image of \( w_\alpha \) is contained in a small neighborhood \( U \) of \( p \) as in Figure 8.2.

We may choose a coordinate chart

\[
\phi : (U, \partial U \cap R) \longrightarrow (\mathbb{C}^n, \mathbb{R}^n)
\]

that satisfies

\[
\begin{aligned}
\phi(p) &= 0, \\
\phi_* J &= i, \quad \text{on } \mathbb{R}^n.
\end{aligned}
\]

Then it is easy to see that \( \phi \circ u_\alpha \) is \( \phi_* J \)-holomorphic.
Exercise 8.4.14  Prove the last statement.

Lemma 8.4.15  \((\partial \phi \circ u_\alpha)/\partial \nu\) is perpendicular to \(\mathbb{R}^n \subset \mathbb{C}^n\), where \(\nu\) is an outward unit normal vector to the boundary \(\partial \Sigma\).

Proof  Choose a Hermitian metric \(h\) in the given conformal class \(j\) and fix cylindrical coordinates \((r, \theta)\) near \(\partial \Sigma\). In other words, \((r, \theta)\) satisfies \(h = \lambda(dr^2 + d\theta^2)\) and \(j \partial/\partial r = \partial/\partial \theta\) in this coordinate system. The equation \(\bar{\partial} J u = 0\) is equivalent to \(\partial u/\partial r + J \partial u/\partial \theta = 0\).

Owing to the boundary condition, \(\partial u_\alpha/\partial \theta\) is tangent to \(TR\) and hence

\[
J \frac{\partial u}{\partial r} \in TR
\]

by the condition \(\phi_* J = i\) on \(\mathbb{R}^n\). By the choice of local chart \(\phi\), this implies

\[
\frac{\partial \phi \circ u_\alpha}{\partial r} \perp \mathbb{R}^n,
\]

which finishes the proof. \(\square\)

Now the proof of Theorem 8.4.13 follows from the following exercise.

Exercise 8.4.16  Define \(f_\alpha : D^2 \to \mathbb{R}\) by \(f_\alpha(z) = |\phi \circ u_\alpha|^2\). Prove \(\partial f_\alpha/\partial \nu|_{\partial D^2} = 0\). Also prove that \(\Delta f_\alpha \geq 0\), assuming that \(\phi_* J\) is \(C^2\)-close to the standard complex structure \(i\) on \(\mathbb{C}^n\) on the image of \(\phi \circ u_\alpha\).

With this exercise, the strong maximum principle implies that \(f_\alpha\) must be constant. But this contracts to the condition \(E(u_\alpha) > 0\), which implies that \(E(\phi \circ u_\alpha) > 0\). This finishes the proof of Theorem 8.4.13. \(\square\)

Remark 8.4.17  We will see that getting a global energy bound for maps \(u : D \to (M, J)\) with \(J\) compatible to a symplectic form \(\omega\) and with a Lagrangian boundary condition comes from the topology of maps by Lemma 7.2.3. Such a topological bound is not available for the totally real case in general.

8.5 Boundary regularity of weakly J-holomorphic maps

Discussion of the weakly \(J\)-holomorphic maps on a closed surface \(\Sigma\) is easier to handle and can be subsumed in the study of the surfaces with a boundary. Therefore we will focus on the case with a boundary in the following discussions.

Let \((N, J, g)\) be an almost-Hermitian manifold and \(R \subset N\) be a totally real submanifold of dimension \(n\) with respect to \(J\).

We introduce the notion of weak solutions of \(J\)-holomorphic maps with totally real boundary condition
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\[ \bar{\partial}_J u = 0, \quad u(\partial \Sigma) \subset R. \quad (8.5.33) \]

For this purpose, we need to use a Hermitian metric \( g_0 \) such that \( TR \perp_{g_0} JTR \) which can be chosen by Lemma 8.3.1. By the Nash embedding theorem, we assume that \((N, g_0)\) is isometrically embedded into \( \mathbb{R}^N \) and consider the map \( u \) and a vector field \( \xi \) along \( u \) as maps from \( \Sigma \) into \( \mathbb{R}^N \). In particular, we can talk about an \( L^1 \) map into \( M \) as an \( L^1 \) map into \( \mathbb{R}^N \) with its values lying in \( M \) almost everywhere on its domain. Then we will use the Levi-Civita connection of the metric \( g_0 \), unlike in other previous sections.

With this understood, motivated by the definition from (Ye94), we give the following definition.

**Definition 8.5.1** We say that a \( W^{1,2} \) map \( u : (\Sigma, j) \to (N, J) \), which is \( L^1 \) on \( \partial \Sigma \), mapping almost all points in \( \partial \Sigma \) into \( R \) is a weak solution of (8.5.33) (with respect to \( g_0 \)) if

1. it satisfies \( \bar{\partial}_J u = 0 \) almost everywhere on \( \Sigma \)
2. it satisfies

\[
\int_{\Sigma} \langle J du, \nabla X \rangle = 0 \quad (8.5.34)
\]

whenever \( X \) is a smooth vector field along \( u \) and for \( z \in \partial \Sigma, X(z) \perp_{g_0} R \) and \( \int_{\partial \Sigma} \langle \nabla X, \nabla X \rangle < \infty \).

We call \( u \) a weakly J-holomorphic map with a Neumann boundary condition on \( R \) (with respect to \( g_0 \)).

**Lemma 8.5.2** Any \( C^1 \) solution of (8.5.33) satisfying (8.5.34) satisfies the Neumann boundary condition in the classical sense.

**Proof** To prove the boundary condition (8.5.34), let \( X \) be any (smooth) vector field on \( D \) with compact support in \( \bar{D} \) and perpendicular to \( R \) on the boundary with respect to \( g_0 \). In the polar coordinates of \( D \), the equation \( \bar{\partial}_J u = 0 \) becomes

\[
\frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} = 0.
\]

We first consider the case in which \( u \) is \( C^2 \). In this case, using Stokes’ formula, we compute

\[
\int_D \langle J du, \nabla X \rangle = \int_D \left( \left( \frac{\partial u}{\partial r}, \frac{DX}{\partial r} \right) + \frac{1}{r^2} \left( \frac{\partial u}{\partial \theta}, \frac{DX}{\partial \theta} \right) \right) r \, dr \, d\theta
\]

\[
= \int_D \left( \left( \frac{1}{r} \frac{\partial u}{\partial r}, \frac{DX}{\partial r} \right) - \left( \frac{\partial u}{\partial r}, \frac{1}{r} \frac{DX}{\partial \theta} \right) \right) r \, dr \, d\theta
\]
\[= - \int_D \left( \frac{D}{\partial r} \frac{\partial u}{\partial \theta}, X \right) dr d\theta + \int_D \left( \frac{D}{\partial \theta} \frac{\partial u}{\partial r}, X \right) dr d\theta\]

\[= - \int_{\partial D} \left( \frac{\partial u}{\partial \theta}, X \right) d\theta = \int_{\partial D} \left( \frac{\partial u}{\partial r}, JX \right) d\theta.\]

Here we used the integration by parts for the third identity, the fact that the Levi-Civita connection is torsion-free, which results in the identity

\[\frac{D}{\partial \theta} \frac{\partial u}{\partial r} = \frac{D}{\partial r} \frac{\partial u}{\partial \theta}\]

for the fourth equality.

Therefore, if \(u\) satisfies (8.5.34), we obtain

\[\int_{\partial D} \left( \frac{\partial u}{\partial r}, JX \right) d\theta = 0\]

for all smooth \(X \perp TL\), or equivalently \(JX\) tangent to \(L\) along \(\partial D\). This implies that \(\partial u/\partial r \perp L\), which finishes the proof. For a \(C^1\) map \(u\), we approximate \(u\) by a sequence of \(C^2\) maps converging to \(u\) in \(C^1\) and apply the limiting argument, whose details we leave as an exercise. \(\square\)

**Exercise 8.5.3** Fill in the details of the limiting argument left out at the end of the above proof.

To translate the meaning of the weak boundary condition (8.5.34) into a more standard distributional boundary condition in \(\mathbb{R}^{2n}\), we first write the \(J\)-holomorphic equation with a Neumann boundary condition in terms of the moving frame of the Levi-Civita connection along \(R\). (A similar computation was carried out by Parker and Wolfson in the appendix of (PW93).)

Choose an orthonormal frame of \(N\)

\[\{e_1, e_2, \ldots, e_n, f_1, \ldots, f_n\}\]  (8.5.35)

such that \(f_j = J e_j\) and \(\{e_1, e_2, \ldots, e_n\}\) spans \(TR \subset TN\). Then

\[u_j = \frac{1}{\sqrt{2}} (e_j - if_j), \quad j = 1, \ldots, n\]

form a unitary frame of \(T^{(1,0)}N\) such that

\[\text{Re } u_j|_R = \frac{1}{\sqrt{2}} e_j, \quad j = 1, \ldots, n.\]  (8.5.36)
Let
\[ \{ \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \} \] (8.5.37)
be the dual frame of (8.5.35). The complex valued 1-forms
\[ \theta^j = \frac{1}{\sqrt{2}}(\alpha^j + i\beta^j), \quad j = 1, \ldots, n \]
form a unitary frame of \((T^*_\mathbb{C}M)^{(1,0)}\) that is dual to the unitary frame \(\{u_1, \ldots, u_n\}\).

An almost-Hermitian connection \(\nabla\) can be expressed by a locally defined, matrix-valued complex 1-form \(\omega = (\omega^i_j)_{i,j=1,2,\ldots,n}\).

Now let \(\{\eta^1, \eta^2\}\) be an orthonormal coframe on \(\Sigma\). Then \(\phi = \eta^1 + i\eta^2\) is a \((1,0)\)-form for \(j\) on \(\Sigma\). The first structure equation of \(\{\eta^1, \eta^2\}\) is
\[ d\phi = -i\rho \wedge \phi = -i\rho \overline{\phi} \wedge \phi, \] (8.5.38)
where \(\rho = \rho_\phi \phi + \rho_{\overline{\phi}} \overline{\phi}\) is the connection form of the Levi-Civita connection for the coframe \(\{\eta^1, \eta^2\}\). The Gauss curvature \(K\) of the metric \(h\) is defined by the second structure equation
\[ d\rho = -\frac{i}{2} K \phi \wedge \overline{\phi}. \]

Then the \(J\)-holomorphicity of \(u\), which is equivalent to the property that \(du\) preserves types of (complex) tangent vectors, is nothing but
\[ u^* \theta^j = a^j \phi \] (8.5.39)
for all \(j = 1, \ldots, n\) (see Lemma 7.3.1), where \(a^j\) are complex-valued functions on \(\Sigma\). We also have the energy formula
\[ |du|^2 = 2 \sum_{j=1}^{n} |a^j|^2. \]
Since we assume that \(du\) is in \(L^2\), so are \(a^j\). Using the continuity of \(u\) and choosing a sufficiently small neighborhood of the given point \(u(z) \in Y\) with \(z \in \partial \Sigma\), we may assume that the frames and hence the \(a^j\) are defined globally in the neighborhood.

By our definition of \(\partial u\) in Chapter 7,
\[ (\partial u)' = \sum_i u_i^* \theta^i \cdot u_i = \sum_i (a_i \phi) \cdot u_i. \]
The equation \(\overline{\partial} (\partial u)' = 0\) from Theorem 7.3.4 is translated into
\[ 0 = \sum_i \overline{\partial}(a_i \phi) \cdot u_i + \sum_j a^j \phi \wedge \sum_i a_i \omega^j \cdot u_i. \] (8.5.40)
We compute
\[ \overline{\partial}(a^i \phi) = \overline{\partial} a^i \wedge \phi - i \rho \overline{\phi} \wedge \phi \]
using (8.5.38). Following the standard notational convention, we denote
\[ \nabla' a^i := da^i \cdot \phi, \quad \nabla'' a^i := da^i \cdot \overline{\phi}. \]
We write
\[ \omega^i_{\ jk} = \omega^i_{\ jk} \theta^k + \omega^i_{\ jk} \overline{\theta}^k. \]
Substituting this into (8.5.40) and using (8.5.39), we rewrite the above equation into
\[ \nabla'' a^i - i \rho \overline{\phi} a^i - \omega^i_{\ jk} a^j a^k = 0. \quad (8.5.41) \]
On the other hand, it follows from (8.5.36) that \( \text{Re} \theta_j \big|_{TR} \) has real values and, by the Neumann boundary condition, we have
\[ du \left( \frac{\partial}{\partial \nu} \right) \in \text{span} \{ f_j \}. \quad (8.5.42) \]
But, since \( du \) is a real one-form, we have
\[ du = \sum_j u^* \theta^j \cdot u_j + \sum_k u^* \overline{\theta}^k \cdot \overline{u}_k = \sum_j (a^j \phi \cdot u_j + \overline{a}^j \overline{\phi} \cdot \overline{u}_j) \]
and hence
\[ du \left( \frac{\partial}{\partial \nu} \right) = \sum_j a^j \phi \left( \frac{\partial}{\partial \nu} \right) \cdot u_j + \overline{a}^j \overline{\phi} \left( \frac{\partial}{\partial \nu} \right) \cdot \overline{u}_j. \]
At this point, we choose isothermal coordinates \((r, \varphi)\) so that \( \partial/\partial \nu = \partial/\partial r \) and \( \phi = \mu (dr + i \, d\varphi), \quad \mu = \mu(r, \varphi) > 0. \)
Then \( \phi(\partial/\partial \nu) = \overline{\phi}(\partial/\partial \nu) = \mu \) are real functions. Therefore we obtain
\[ du \left( \frac{\partial}{\partial \nu} \right) = \mu \sum_j (a^j \cdot u_j + \overline{a}^j \cdot \overline{u}_j) = \mu \left( \sum_j (a^j + \overline{a}^j) e_j - i(a^j - \overline{a}^j) f_j \right). \]
The boundary condition (8.5.42) then is equivalent to
\[ a^j + \overline{a}^j = 0 \]
i.e., \( a^j \) are purely imaginary on \( \partial \Sigma \) for all \( j = 1, \ldots, n. \)
We summarize the above calculation as follows.

**Proposition 8.5.4** Let \( \{ \theta^i \} \) be a moving frame adapted to \( R \) as above and let \( u \) be a smooth J-holomorphic map with a Neumann boundary condition along \( R \) on \( \partial \Sigma \). Denote \( u^* \theta^j = a^j \phi \) for a complex-valued function \( a^j \) for each \( j = 1, \ldots, n \). Then we have

\[
\Delta \nabla (\partial u) = 0,
\]

which is equivalent to

\[
\begin{align*}
\nabla'' a^i - i \rho_\bar{\phi} a^i - \omega_{j\ell} a^j \bar{a}^\ell = 0, \\
a^i \text{ purely imaginary on the boundary for } i = 1, \ldots, n.
\end{align*}
\]

(8.5.43)

The following regularity theorem for the weakly J-holomorphic map \( u : \Sigma \to M \) with Neumann boundary condition is the main theorem of this section.

**Theorem 8.5.5** Any weakly J-holomorphic map \( u : \Sigma \to M \) with a Neumann boundary condition along \( R \) on \( \partial \Sigma \) that is \( W^{1,p} \) for some \( p \geq 4 \) or is \( C^\epsilon \) for some \( \epsilon > 0 \) is smooth.

**Proof** First note that the Sobolev embedding \( W^{1,p} \hookrightarrow C^\epsilon \) for \( p > 2 \) and \( 0 < \epsilon < 1 - 2/p \) implies that \( u \) is a \( C^\epsilon \) solution of (8.5.33) with the weak boundary condition (8.5.34). Since \( u \) is continuous, we can localize the study of regularity. Because the interior regularity result is easier, we will focus on the boundary regularity.

Let \( \{ \theta^i \} \) be a moving frame of \( (T^*_\Sigma \mathbb{C}M)^{(1,0)} \) adapted to \( R \) as above. Since \( u \in W^{1,p} \), there exists a sequence of smooth maps \( u_\alpha \) converging to \( u \) in the \( W^{1,p} \) topology by the Schoen–Uhlenbeck strong approximation theorem (ScU83). Then we have

\[
\int_\Sigma |\overline{\partial} J u_{\alpha} |^2 \to 0
\]

and

\[
\int_\Sigma \langle \partial u_\alpha, (\bar{\partial}^\nabla)^* X \rangle \to \int_\Sigma \langle \partial u, (\bar{\partial}^\nabla)^* X \rangle = 0
\]

as \( \alpha \to \infty \) for any given vector field \( X \) along \( u \) such that \( X(z) \) is tangent to \( Y \) and \( \int_\Sigma \langle \nabla X, \nabla X \rangle < \infty \).

If we write

\[
u^*_\alpha \theta^j = a^j_\alpha \phi + b^j_\alpha \bar{\phi},
\]

these, combined with (8.5.39), imply \( b^j_\alpha \to 0 \) in \( L^p \) and \( a^j_\alpha \) converges to \( a^j \) in \( L^p \), and \( a = \{ a^j \} \) is a weak solution of (8.5.43). Note that, in (8.5.43), the terms

\[
- i \rho_\bar{\phi} a^i - \omega_{j\ell} a^j \bar{a}^\ell
\]

(8.5.44)
8.6 The removable singularity theorem

lie in $L^{p/2}$. Here $p/2 \geq 2$ by the hypothesis $p \geq 4$. We also note that $\nabla''$ has the form $\nabla'' = \overline{\partial} + \Gamma$ for a zeroth-order operator $\Gamma$ lying in $C^\epsilon$. It follows from these that the equation (8.5.43) has the form

$$
\begin{cases}
\overline{\partial}a^i = f^i, \\
a^j \text{ purely imaginary on the boundary}
\end{cases}
$$

with $f^i \in L^p$. But this forms a first-order elliptic boundary-value problem with the same symbol map as $\overline{\partial}$. A standard regularity result (Mo66) (or a special case of Theorem 8.3.5) then implies the inequality

$$
\|da\|_{1,p,K} \leq C_2\|f\|_{0,p,D} \leq C_3(\|a\|_{0,p,D} + \|a\|_{0,p/2,D})
$$

after combination with (8.5.43).

Therefore $du$ lies in $W^{2,p} \hookrightarrow C^\epsilon$ and hence $u \in C^{1,\epsilon}$. Further regularity then follows by regarding (8.5.43) as a linear equation of $a^j$ whose coefficients are Hölder continuous and that satisfies the imaginary boundary condition and then applying standard bootstrap arguments. This finishes the proof. □

According to this theorem, the study of local regularity boils down to obtaining $W^{1,p}$ estimates for $J$-holomorphic maps with $p \geq 4$, especially for those with finite area or harmonic energy.

8.6 The removable singularity theorem

In this section, we prove the following removable singularity theorem. Denote by $D$ either the open unit disc or the semi-disc with $\partial D = (-1, 1)$ as before. Let $(M, \omega, J)$ be almost-Kähler, not necessarily compact, and let $L \subset M$ be a compact Lagrangian submanifold without a boundary. We assume that $(M, \omega)$ is tame and so carries a compatible $J$ such that $(M, J, g)$ with $g = \omega(\cdot, J\cdot)$ is tame in the sense of Definition 8.7.7.

**Theorem 8.6.1** Let $L \subset (M, \omega)$ be a Lagrangian submanifold. Let $u : D\setminus\{0\} \to M$ be a $J$-holomorphic map satisfying $u(\partial D) \subset L$ and $\text{Im } u \subset K \subset M$ for some compact subset $K$. If $\int_{D\setminus\{0\}} |du|^2 < \infty$, then $u \in W^{1,p}$ for some $p > 2$.

**Corollary 8.6.2** Let $u : D\setminus\{0\} \to M$ be as in Theorem 8.6.1. Then $u$ smoothly extends to $D$.

**Proof** By the regularity theorem, Theorem 8.5.5, it suffices to prove that the extended $W^{1,p}$ map $u$ on $D$ is weakly $J$-holomorphic. Obviously we have

$$
\int_D |\overline{\partial}Ju|^2 = 0
$$
since \( u \) is smooth \( J \)-holomorphic on \( D^2 \setminus \{0\} \) and hence \( \overline{\partial}_J u = 0 \) almost everywhere.

To prove the boundary condition (8.5.34), let \( X \) be any (smooth) vector field on \( D \) with compact support in \( \partial D \) and perpendicular to \( R \) on the boundary with respect to \( g_0 \). Since \( u \) lies in \( W^{1,p} \) and \( X \) lies in \( W^{1,2} \), we have

\[
\int_D \langle J du, \nabla X \rangle = \lim_{r \to 0} \int_{D \setminus (D(r) \setminus D)} \langle J du, \nabla X \rangle.
\]

Let \((x, y)\) be the standard coordinate on \( \mathbb{H} \) and assume that \( \partial D \subset \partial \mathbb{H} \).

Using the equation \( \overline{\partial}_J u = 0 \) on \( D \setminus (D(r) \setminus D) \) and integration by parts, we compute

\[
\int_{D \setminus (D(r) \setminus D)} \langle J du, \nabla X \rangle = \int_{D \setminus (D(r) \setminus D)} \left( \frac{\partial u}{\partial y} X - \frac{\partial u}{\partial x} DX \right) \, dx \, dy
\]

which follows from Corollary 7.3.3. For the last equality, we use the fact that \( X \perp g_0 R \) and \( \frac{\partial u}{\partial x} \) is tangent to \( R \) and so \( \left\langle \frac{\partial u}{\partial x}, X \right\rangle = 0 \). Finally we have

\[
\left| \int_{\partial D(r)} \left( \frac{\partial u}{\partial x} X - \frac{\partial u}{\partial y} X \right) \, dx - \left\langle \frac{\partial u}{\partial y}, X \right\rangle \, dy \right| \leq 2\|X\|_C \, L(u|_{\partial D(r)})
\]

which converges to 0 by Courant–Lebesgue lemma below. This finishes the proof.

For the proof of Theorem 8.6.1, we start with the following useful lemma. This version of the lemma for the semi-disc is taken from (Oh92).

**Lemma 8.6.3 (Courant–Lebesgue lemma)** Let \( 0 < \sigma < 1 \) be any given constant. Consider a map \( u \in W^{1,2}(D, M) \) or \( u \in W^{1,2}((D, \partial D), (M, L)) \). Suppose that \( \int_{D(\sigma)} |du|^2 \leq \Gamma \) for some \( \Gamma > 0 \). Then there exists \( \rho \in [\sigma^2, \sigma] \) such that

\[
L(u|_{x=\rho})^2 \leq \frac{C \Gamma}{\log(1/\rho)}, \quad C = C(\Gamma)
\]

where \( L \) is the length function.
\[ L(\gamma)(s) = \int |\gamma'| dt. \]

We note that for an element \( u \in W^{1,2} \) the length \( L(u|_{|x| = \rho})^2 \) is defined almost everywhere but not necessarily everywhere over \( \rho \). Applying this lemma to the sequence \( \ldots < \sigma^n < \sigma^{n-1} < \ldots < \sigma < 1 \), it in particular implies that we get a sequence \( \rho_i \) for which the length of the curve \( u|_{|x| = \rho} \) converges to 0.

**Proof** Again we will only look at the semi-disc case. We compute
\[
\int_{\sigma^2}^{\sigma} \frac{L(u|_{|x| = r})^2}{r} dr = \int_{\sigma^2}^{\sigma} \frac{1}{r} \left( \int_0^{\pi} \left| \frac{\partial u}{\partial \theta} \right|_{|x| = r}^2 d\theta \right) dr \\
\leq \int_{\sigma^2}^{\sigma} \frac{1}{r} \left( \int_0^{\pi} \left| \frac{\partial u}{\partial \theta} \right|_{|x| = r}^2 d\theta \cdot \pi \right) dr \\
= \pi \int_{\sigma^2}^{\sigma} \int_0^{\pi} \frac{1}{r} \left| \frac{\partial u}{\partial \theta} \right|_{|x| = r}^2 r dr d\theta \\
\leq \pi \int_{D(\sigma) \setminus D(\sigma^2)} |du|^2 \leq \pi \Gamma.
\]
written in polar coordinates. Here we used Hölder’s inequality for the first inequality. Therefore \( L(u|_{|x| = r})^2 \) is defined a.e. on \([\sigma^2, \sigma]\). If \( L(u|_{|x| = r})^2 > C \Gamma / \log(1/r) \) a.e. on \([\sigma^2, \sigma]\) for some \( C > 0 \), then we would have
\[
\int_{\sigma^2}^{\sigma} \frac{L(u|_{|x| = r})^2}{r} dr > \int_{\sigma^2}^{\sigma} \frac{C \Gamma}{r \log(1/r)} dr = C \Gamma \log 2.
\]
Therefore if we choose \( C \log 2 > \pi \), it cannot hold that \( L(u|_{|x| = r})^2 > C \Gamma / \log(1/r) \) a.e. on \([\sigma^2, \sigma]\), and hence the proof has been completed. \( \square \)

Now we are ready to give the proof of Theorem 8.6.1.

**Proof of Theorem 8.6.1** Note that by Fatou’s Lemma
\[
\lim_{r \to 0} \int_{D(\sigma) \setminus D(r)} |du|^2 \to \int_{D(\sigma) \setminus \{0\}} |du|^2 < \infty.
\]
So we can make \( \int_{D(r_0) \setminus \{0\}} |du|^2 < \epsilon \) by choosing a sufficiently small \( r_0 > 0 \) for the constant \( \epsilon \) appearing in the \( \epsilon \)-regularity theorem.

**Lemma 8.6.4** Let \( U \) be a given Darboux neighborhood of \( L \). If we choose \( r_1 < r_0 \) sufficiently small, then
\[ \text{Im}(u(D(r_1) \setminus \{0\})) \subset U. \]

**Proof** Suppose to the contrary that there exists a sequence \( z_i \in D \setminus \{0\} \) such that \( |z_i| =: r_i \to 0 \), but \( u(z_i) \notin U \).

Since \( K \cap L \) is compact, we know that there exists \( \delta > 0 \) such that \( \text{dist}(L \cap K, M \setminus U) \geq \delta > 0 \). Therefore, \( \text{dist}(u(z_i), L) \geq \delta \) for all \( i \). This \( \delta \) is
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independent of \( u \) but depends only on \( U \cap K, J, \omega \) and \( L \). On the other hand, by the Courant–Lebesgue lemma, we obtain some sequences \( \{ s_i \} \) and \( \{ s'_i \} \) such that \( s_i < r_i < s'_i \) and

\[
L(u|_{|z|=s_i}), L(u|_{|z|=s'_i}) < \frac{\delta}{4}.
\]

Then we have the inequality

\[
\max_{|z|=s_i, s'_i} \text{dist}(u(z), L) \leq \frac{\delta}{4}
\]

because of the boundary condition \( u|_{\partial D} \subset L \).

Now \( \text{Im} u \cap B(u(z_i), \delta/2) \) defines a proper \( J \)-holomorphic curve in \( B(u(z_i), \delta/2) \) with \( u(z_i) \) at the center of the \( B(u(z_i), \delta/2) \). Therefore by the monotonicity formula, Proposition 8.7.10 in the next section, we have

\[
\text{Area} \left( \text{Im} u \cap B \left( u(z_i), \frac{\delta}{2} \right) \right) \geq C \left( \frac{\delta}{2} \right)^2,
\]

where \( C \) depends only on \((K, J, \omega)\). This would give rise to a contradiction if we choose \( \epsilon \) so that \( 0 < \epsilon < C(\delta/2)^2 \) because then

\[
\text{Area} \left( \text{Im} u \cap B \left( u(z_i), \frac{\delta}{2} \right) \right) \leq \text{Area}(u) \leq \epsilon.
\]

This finishes the proof.

We go back to the proof of Theorem 8.6.1. By conformally expanding \( D(r_1) \) to \( D \) and considering the corresponding conformal reparameterization of \( u : D(r_1) \setminus \{ 0 \} \rightarrow M \), we may assume without any loss of generality that

\[
\int_{D \setminus \{ 0 \}} |du|^2 < \epsilon.
\]

Now fix a one-form \( \beta \) defined on \( U \) so that \( \omega = -d\beta \) and \( \beta|_L = 0 \). For example, the Liouville one-form of the cotangent bundle restricted to \( U \) is such a form. Consider the annulus

\[
A_m = \{ z \in D \mid 0 < r_m \leq |z| \leq 1 \}.
\]

See Figure 8.3. Since \( u \) is \( J \)-holomorphic, we derive

\[
\frac{1}{2} \int_{A_m} |du|^2 = \int_{A_m} u^* \omega = -\int_{A_m} u^* d\beta = -\int_{S(1)} u^* \beta + \int_{S(r_m)} u^* \beta \pm \int_{A_m \cap \{ \text{Im} z = 0 \}} u^* \beta = -\int_{S(1)} u^* \beta + \int_{S(r_m)} u^* \beta + 0.
\]  

(8.6.45)
The third term in the last line vanishes by the property $\beta_L \equiv 0$. (This is the place where the Lagrangian property of $L$ has some advantage over the totally real boundary condition. The case of a totally real boundary condition requires a more sophisticated estimate, as illustrated in \((\text{Ye94})\).)

Note that, since $|du|^2$ is non-negative and $\int_{D\{0\}} |du|^2 < \infty$, Fatou’s lemma implies that
\[
\lim_{m \to \infty} \frac{1}{2} \int_{A_m} |du|^2 = \frac{1}{2} \int_{D\{0\}} |du|^2.
\]
On the other hand,
\[
\left| \int_{S(r_m)} u^* \beta \right| \leq \int_0^{2\pi} \left| \beta \left( \frac{\partial u}{\partial \theta} \right)_{|z|=r_m} \right| d\theta
\]
and
\[
\int_0^{2\pi} \left| \beta \left( \frac{\partial u}{\partial \theta} \right)_{|z|=r_m} \right| d\theta \leq C||\beta||_{C^0} \int_0^{2\pi} \left| \frac{\partial u}{\partial \theta} \right| d\theta = C||\beta||_{C^0} \cdot L(u_{|z|=r_m}).
\]
Recall that we have chosen $r_m$ so that $L(u_{|z|=r_m}) \to 0$ at the beginning of the proof. Therefore we have proved $\lim_{m \to \infty} \int_{S(r_m)} u^* \beta = 0$, and hence obtain
\[
\frac{1}{2} \int_{D\{0\}} |du|^2 = - \int_{S(1)} u^* \beta.
\]
But, since $|u^* \beta| \leq ||\beta||_{C^0} |\partial u/\partial \theta|$, we obtain
\[
\frac{1}{2} \int_{D\{0\}} |du|^2 \leq C||\beta||_{C^0} \cdot L(u_{|z|=1}).
\]

Note that $C > 0$ is independent of any specific $J$-holomorphic curve $u$ and this inequality holds as long as $\epsilon > 0$ is chosen as before. In particular, this can be applied to the conformal reparameterization $u_r$ defined by
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of the map $u_{|0<|z|<r}$ to the unit disc $D$: namely, we have

$$\frac{1}{2} \int_{D\setminus\{0\}} |du_r|^2 \leq C||\beta||_{C^0} L(u_r)|_{z=1}). \quad (8.6.46)$$

But, by the conformal invariance of $E$, we have

$$\frac{1}{2} \int_{D(r)\setminus\{0\}} |du|^2 = \frac{1}{2} \int_{D\setminus\{0\}} |du_r|^2,$$

while the length transforms as

$$L(u_r)|_{z=1}) = \int_0^\pi \left| \frac{\partial u_r}{\partial \theta} \right|_{|z|=1} d\theta \quad = \int_0^\pi r \cdot \left| \frac{\partial u}{\partial \theta} \right|_{|z|=r} d\theta = r \cdot L(u)|_{z=r}). \quad (8.6.47)$$

Now we denote

$$E(r) = \frac{1}{2} \int_{D(r)\setminus\{0\}} |du|^2,$$

$$L(r) = L(u)|_{z=r}),$$

$$A(r) = \text{Area}(u(D(r))).$$

Since $u$ is $J$-holomorphic we have the identity

$$E(r) = A(r) = \text{Area}(u_{|0<|z|<r}). \quad (8.6.48)$$

By combining (8.6.46)–(8.6.48), we obtain

$$A(r) \leq C' \cdot r \cdot L(r),$$

with the constant $C' = C||\beta||_{C^0}$. Therefore we have obtained

$$A(r) \leq C' \cdot r \cdot \frac{dA(r)}{dr} \iff E(r) \leq C' \cdot r \cdot \frac{dE(r)}{dr},$$

which is in turn equivalent to

$$1 \leq C' \cdot r \cdot \frac{1}{E(r)} \frac{dE(r)}{dr}$$

with $r_m < r < 1$ for all $m$. Upon dividing by $C' \cdot r$ and integrating this, we get

$$\int_r^1 \frac{1}{C' \cdot s} ds \leq \log(E(1)) - \log(E(r)) = \log \left( \frac{E(1)}{E(r)} \right).$$

Therefore we obtain

$$\log \left( \frac{E(1)}{E(r)} \right) \geq \log r^{-1/C'}$$
and so
\[ E(r) \leq E(1)r^\lambda, \]
where \( \lambda := 1/C' \). This proves
\[
\int_{D(\|z\| \leq r)} |du|^2 \leq E(1) \cdot r^\lambda. \tag{8.6.49}
\]
On substituting this into the energy density estimate, Proposition 8.4.6, we obtain
\[
|du(z)|^2 \leq \frac{C'}{|z|^2} E(1)|z|^4 = C' \cdot E(1) \cdot |z|^{4-2} \tag{8.6.50}
\]
with \( \lambda > 0 \) for all \( z \) with \( |z| \leq \frac{1}{4} \). Therefore
\[
|du(z)|^p \leq C''|z|^{(\lambda - 2)/2} = C''|z|^p(\lambda/2 - 1)
\]
for \( C'' = C' E(1) \).

We now want to choose \( p \) so that \( |du(z)|^p \) is integrable. But
\[
\int_0^1 \int_0^{2\pi} |z|^{p(\lambda/2 - 1)} r^2 \, dr \, d\theta = 2\pi \int_0^1 r^{p(\lambda/2 - 1) + 1} \, dr = \frac{2\pi}{p(\lambda/2 - 1) + 2} - \lim_{r \to 0} \frac{2\pi}{p(\lambda/2 - 1) + 2}
\]
This limit is finite if we can choose \( p \) so that \( p(\lambda/2 - 1) + 2 > 0 \).

Finally, we want to choose \( p \) so that
\[
p > 2 \quad \text{and} \quad p\left(\frac{\lambda}{2} - 1\right) + 2 > 0.
\]
If \( \lambda/2 - 1 \geq 0 \), any choice \( p > 2 \) will do. On the other hand, if \( \lambda/2 - 1 < 0 \), we choose \( p \) so that
\[
2 < p < \frac{2}{1 - \lambda/2}
\]
which is possible because \( 2/(1 - \lambda/2) > 2 \) in that case. This finishes the proof. \( \square \)

**Exercise 8.6.5** Modify the above proof and provide the proof of the removal theorem of interior singularity. (Hint. Use the ordinary Darboux theorem instead of the Darboux–Weinstein theorem. But where would you choose the Darboux neighborhood?)

In the remainder of this section, we will give a useful translation of the inequality (8.6.50) into exponential decay estimates of \( J \)-holomorphic maps.
with finite energy defined on an open Riemann surface with strip-like ends with a boundary lying on $L$.

Consider a $J$-holomorphic map

$$u : (-\infty, 0] \times [0, 1] \to (M, L)$$

on a semi-strip $(-\infty, 0] \times [0, 1]$ with finite energy $E_J(u) < \infty$ and with a bounded image satisfying the boundary condition

$$u((-\infty, 0] \times \{0, 1\}) \subset L.$$

By composing with the conformal isomorphism

$$\varphi : (-\infty, 0] \times [0, 1] \cong (D(1), \partial D(1)) \setminus \{0\}, \quad \varphi(\tau, t) = e^{2\pi(\tau + it)}$$

we consider the map $\tilde{u}(z) = u(\varphi^{-1}(z))$. We have the relation

$$|du|_{cyl} = 2\pi|z||d\tilde{u}|_{polar}.$$ 

By the conformal invariance, we have $E_J(u) = E_J(\tilde{u}) < \infty$. Applying (8.6.50) to $\tilde{u}$, we obtain

$$|du|_{cyl}(\tau, t) \leq C|z|^{(1/2 - 1) + 1} = Ce^{-2\pi|t|/\lambda}.$$

This gives rise to the following exponential decay estimate.

**Theorem 8.6.6 (Exponential decay)** Let $K \subset M$ be a compact domain. Consider a $J$-holomorphic map $u : (-\infty, 0] \times [0, 1] \to (M, L)$ with finite energy $E_J(u) < \infty$ and a bounded image that satisfies the boundary condition

$$u((-\infty, 0] \times \{0, 1\}) \subset L.$$

Then there must exist constant $C$, $\delta$, $R > 0$, depending only on $(K, L)$, $E_J(u)$ and the diameter of the image of $u$, such that

$$|du|_{cyl}(\tau, t) \leq Ce^{-\delta|t|} \quad (8.6.51)$$

for all $\tau \leq -R$.

**Exercise 8.6.7** Prove the $C^k$ exponential decay estimate: there exists $C(k)$, $R(k) > 0$ depending on $k$ such that

$$|\nabla^k u|(\tau, t) \leq C(k)e^{-\delta|t|} \quad (8.6.52)$$

for all $\tau \leq -R(k)$. 
8.7 Isoperimetric inequality and the monotonicity formula

We recall Corollary 7.2.6, which states that, for an almost-Kähler triple $(M, \omega, J)$, “the image of any J-holomorphic curve is a minimal surface with respect to the associated metric $g = \omega(\cdot, J \cdot)$”.

We first make the above statement in quotes more precise. To do this, we need to study the structure of the image of J-holomorphic curves. The following proposition is an immediate consequence of the similarity principle, Theorem 8.8.2 in Section 8.8. See Corollary 8.8.6 and Theorem 8.8.9 in particular. (See also (Mc87) and (Sik94) for such theorems.)

**Proposition 8.7.1** Let $(N, J)$ be any almost-complex manifold of Hölder class $C^\epsilon$ with $\epsilon > 0$ and let $\Sigma$ be either a closed Riemann surface or one with a boundary.

1. Then, for any two different J-holomorphic maps $f, g : \Sigma \to N$, the set of the points $z$ satisfying $f(z) = g(z)$ has no cluster points.
2. Assume that $J$ is of class $C^1$ and let $f : \Sigma \to N$ be non-constant J-holomorphic. Then the set of critical points of $f$ is isolated.

The same statement holds also for a J-holomorphic map $f : (\Sigma, \partial \Sigma) \to (N, R)$ for a surface with nonempty boundary $\Sigma$ with totally real boundary condition $R$.

One immediate consequence of this proposition is that when $\Sigma$ is closed the image of any J-holomorphic map has a finite number of singularities.

**Definition 8.7.2** A J-holomorphic curve $f : \Sigma \to N$ is called somewhere injective if there exists a point $z \in \Sigma$ such that $\#(f^{-1}(f(z))) = 1$ and $df(z) \neq 0$.

The following structure theorem is an important ingredient in the study of pseudoholomorphic curves. We refer readers to (Mc87) for its proof.

**Theorem 8.7.3** (Structure theorem (Mc87)) Let $\Sigma$ be a closed Riemann surface. If a J-holomorphic map $f : \Sigma \to N$ is not somewhere injective, then there exists a somewhere-injective map $\tilde{f} : \tilde{\Sigma} \to N$ and a branched holomorphic covering $\phi : \Sigma \to \tilde{\Sigma}$ such that $f = \tilde{f} \circ \phi$.

Now we are ready to describe the structure of the image of somewhere-injective curves.
Definition 8.7.4 Suppose $\Sigma$ is a closed Riemann surface and $f : \Sigma \to N$ is $J$-holomorphic and somewhere injective. Denote $A = \text{Im } f$. We say that a point $x \in A$ is singular if either $\#(f^{-1}(x)) \geq 2$ or $x = f(z)$ for some $z$ with $df(z) = 0$, and regular otherwise.

Corollary 8.7.5 Suppose $\Sigma$ is a compact Riemann surface with boundary $\partial \Sigma$ and $f : \Sigma \to N$ is $J$-holomorphic. Denote $A = \text{Im } f$. Suppose there is a regular point $y \in A$. Then $\#(\text{Sing}(A)) < \infty$ and hence the area of $A$ is well defined for any given metric $g$ on $N$.

Proof The finiteness $\#(\text{Sing}(A)) < \infty$ immediately follows from the compactness of $\Sigma$ and the above Proposition 8.7.1. Since $A$ is a differentiable embedded surface near each regular point and there are only finitely many singular points in $A$, we just define

$$\text{Area } A = \text{Area}(A \setminus \text{Sing } A) = \int_{\Sigma} |f_x \wedge f_y| dx \, dy,$$

where $(x, y)$ are coordinates of $(\Sigma, j)$ and $|f_x \wedge f_y|$ is the area density function of $f$ with

$$|f_x \wedge f_y|^2 = |f_x|^2|f_y|^2 - \langle f_x, f_y \rangle^2.$$

It follows from the compactness of $\Sigma$ and the differentiability of $f$ that the integral is integrable and well defined. $\square$

We recall that, when $g = g_J = \omega(\cdot, J \cdot)$ for a compatible metric, we have $E_J(f) = \text{Area } A$.

With this preparation, we are now ready to state the important geometric properties of $J$-holomorphic curves, the isoperimetric inequality and the monotonicity property. These properties play crucial roles in the study of the removal singularity theorem and compactification of the moduli space of $J$-holomorphic curves.

We start with the well-known isoperimetric inequality and the monotonicity formula for the minimal surface in the Euclidean space. (See (Law80) for example.)

Lemma 8.7.6 (Classical isoperimetric inequality) Let $\Sigma'$ be the solution of the Plateau problem with the given embedded rectifiable Jordan curve $\ell \subset \mathbb{R}^k$. Then there exists a constant $C_1 > 0$ that is independent of $\Sigma'$ such that

$$\text{Area}(\Sigma') \leq C_1^2 \text{length}^2(\ell). \quad (8.7.53)$$
To translate this classical result into those for $J$-holomorphic curves, we follow the flexible geometric set-up used by Sikorav (Sik94).

**Definition 8.7.7** Let $(N, J, g)$ be an almost-complex manifold with a Riemannian metric $g$. We say the triple is tame if the metric $g$ is complete, has bounded curvature and its injectivity radius bounded away from zero, and if $J$ is uniformly continuous with respect to $g$.

We would like to point out that we could require very little regularity for $J$, e.g., $J$ could be in any Hölder class $C^\epsilon$ for $\epsilon > 0$. The following lemma is borrowed from (Sik94).

**Lemma 8.7.8** Suppose $(N, J, g)$ is tame. Then the following statements hold.

1. **(T1)** For all $x \in N$, the exponential map $\exp_x : B(0, r_0) \subset T_x N \to B(x, r_0) \subset N$ is a diffeomorphism.
2. **(T2)** Every loop $\gamma$ in $N$ contained in a ball $B = B(x, r)$ with $r \leq r_0$ bounds a disc in $B$ of area less than $C_3$ length$(\gamma)^2$.
3. **(T3)** On every ball $B = B(x, r_0)$, there exists a symplectic form $\omega_x$ such that $\|\omega_x\| \leq 1$ and $|X|^2 \leq C_4^2 \omega_x(X, JX)$. We call this property the taming property and $\omega_x$ is a (local) taming form of $(N, J)$.

**Proof** (T1) follows immediately from the injectivity radius bound. We now prove (T2). By the hypothesis on $(N, J, g)$, the injectivity radius is positive. Fix a constant $r_0$ smaller than the injectivity radius. Then we can choose an open geodesic ball $B(x, r_0)$ of $g$ together with a taming form $\eta_0$ such that $f(\Sigma) \subset B(x, r_0)$ and

$$g_x = \eta_0(\cdot, J_x \cdot).$$ (8.7.54)

Now let $f : D \to N$ be a $J$-holomorphic disc with $f(D) \subset B(x, r_0)$. Using Lemma 8.7.6, we find a compact surface $S$ with boundary $\partial S = f(\partial \Sigma)$ and

$$\text{Area}_{g_0}(S) \leq C_4^2 \text{length}_{g_0}(\partial S),$$ (8.7.55)

where $g_0 = (\exp_x^{-1})^* g$. On the other hand, by the taming property of $\eta_0$, we have

$$\text{Area}_{g_0}(S) = \int_{\Sigma} \eta_0.$$  

By the contractibility of $B$ (and hence $[f(\Sigma) \# \Sigma] = 0$ in $H_2(N)$) and the multiplicity hypothesis on $f$, we have
\( \int_S \eta_0 = \int_D f^* \eta_0 = \text{Area}_{g_0}(f). \)

Then we have

\[
\frac{g_0}{C^2_2} \leq g \leq C^2_2 g_0
\]

for some constant \( C_2 > 0 \) depending only on \( g \) but independent of the choice of \( x \) and \( r_0 \). Therefore we obtain

\[
\text{Area}_{g_0}(f) \geq \frac{1}{C^2_2} \text{Area}_g(f).
\]

On the other hand, we have

\[
\text{length}^2_{g_0}(\partial S) = \text{length}^2_{g_0}(f(\partial \Sigma)) \leq \frac{1}{C^2_2} \text{length}^2_g(\partial S).
\]

On substituting all these into \((8.7.55)\), we obtain \((T2)\) with \( C^3_2 = \frac{C^3_1}{C^2_2} \).

Finally we prove \((T3)\). Consider the tangent space \( T_xN \) with an endomorphism \( J_x \) with \( J^2_x = -\text{id} \). Choose the constant two-form \( \eta_0 \) on \( T_xN \) so that the bilinear two-form \( \eta_0(\cdot, J_x \cdot) \) on \( T_xN \) is positive definite. Then, by the hypothesis on \( g \) and the uniform continuity of \( J \) in \( g \), \( \eta = (\exp^x_\eta)^* \eta_0 \) is a symplectic form that is well defined and tames \( J \) on a \( B(x, r_0) \) of \( x \), where \( r_0 \) does not depend on \( x \in N \). \( \square \)

Now we assume \((T1)\)–\((T3)\) in Definition 8.7.7 and prove the following general isoperimetric inequality. The idea of using the taming form in the study of pseudoholomorphic curves was originally due to Gromov himself (Gr85).

**Proposition 8.7.9** (Isoperimetric inequality) Assume \( (N, J, g') \) is an almost Hermitian manifold that satisfies \((T1)\)–\((T3)\) and let \( r_0, C_3, C_4 \) as in Lemma 8.7.8. Let \( f: \Sigma \to N \) be a \( J \)-holomorphic map on a compact Riemann surface \( \Sigma \) that is embedded along the boundary \( \partial \Sigma \). Then there exists a constant \( C_5 > 0 \) depending only on \( (N, J, g') \) such that, whenever \( f(\Sigma) \subset B(x, r_0) \),

\[
\text{Area}_{g'}(f(\Sigma)) \leq C^2_5 \text{length}^2_{g'}(\partial f(\Sigma)). \tag{8.7.56}
\]

**Proof** By \((T2)\), \( f(\partial \Sigma) \) bounds a surface \( S \subset B(x, r_0) \) of area less than or equal to \( C^2_3 \text{length}^2_{g'}(\partial f(\Sigma)) \). Using \( \|\omega_\lambda\| \leq 1 \) and contractibility of \( B \), we have

\[
\int \Sigma f^* \omega_\lambda = \int \Sigma \omega_\lambda \leq C^2_3 \text{length}^2_{g'}(\partial f(\Sigma)).
\]

On the other hand the taming property \((T3)\) and \( f \) being \( J \)-holomorphic, we obtain
Area_{g'}(f(\Sigma)) \leq C_4^2 \int_\Sigma f^* \omega_x.

On combining the two inequalities, the previous proposition follows with the choice of constant $C_5 = C_3 C_4$.  

Next we prove the monotonicity, which roughly says that “if the diameter of the image of $J$-holomorphic $f$ is big, then its area must be big”.

**Proposition 8.7.10** (Monotonicity)  Let $(N, J, g')$ and $r_0$ be as in Proposition 8.7.9. Let $f : \Sigma \to N$ be a $J$-holomorphic map with a compact Riemann surface $(\Sigma, j)$ with boundary $\partial \Sigma$. Assume $f(\Sigma) \subset B = B(x, r)$ with $f(\partial \Sigma) \subset \partial B$ and $x \in f(\Sigma)$. Then there exists $C_7 > 0$ with $r_0$ as above such that

\[
\text{Area}_{g'}(f(\Sigma)) \geq C_6^2 r^2.  
\]  

(8.7.57)

**Proof**  Denote $\Sigma_t = f^{-1}(B(x, t))$ and $A(t) = \text{Area}(f|_{\Sigma_t})$. Since $f$ is of class $C^{r+1}$ for some $r > 0$, Sard’s theorem implies that $\Sigma_t$ is a compact subsurface of $\Sigma$, not necessarily connected but with a finite number of connected components, with a $C^1$-smooth boundary $\partial \Sigma_t = f^{-1}(\partial B(x, t))$. We denote by $\ell(t)$ the length of $f(\partial \Sigma_t)$. Then $A(t)$ is an absolutely continuous function and $A'(t) = \ell(t)$ almost everywhere. On the other hand, with the isoperimetric inequality applied to each $\Sigma_t$, we obtain $A(t) \leq C_5^2 \ell(t)^2$ and in turn $A'(t) \geq \sqrt{A(t)/C_5^2}$. This last inequality is equivalent to

\[
\left(\frac{\sqrt{A(t)}}{C_5}\right) \geq \frac{1}{2C_5}
\]

whenever $A(t)$ is differentiable. By integrating this differential inequality over $[0, r]$, we obtain $\sqrt{A(r)} \geq r/(2C_5)$, i.e., $A(r) \geq r^2/(4C_5^2)$. Just choose $C_6 = 1/(2C_5)$. (Compare this proof with that of Theorem 8.6.1.)

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**8.8 The similarity principle and the local structure of the image**

In this section, we introduce a useful local similarity principle that relates the $J$-holomorphic curve equation with a totally real boundary condition to the classical holomorphic curves with a boundary condition expressed totally in complex variables. This principle enables one to reduce many local studies and estimates of $J$-holomorphic curves to those of the classical holomorphic curves. The principle was exploited by Floer, Hofer and Salamon (FHS95) and by the present author (Oh97a) for the study of interior properties and for
that of boundary properties, respectively. We will focus on the boundary case, leaving the interior one out since the interior case is clearly subsumed into the boundary case.

We define $J(2n)$ to be the set of automorphisms $I : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfying $I^2 = -\text{id}$. We call any such $I$ a linear complex structure on $\mathbb{R}^{2n}$.

**Definition 8.8.1** Let $I : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be an automorphism with $I^2 = -\text{id}$. An $n$-dimensional subspace $V \subset \mathbb{R}^{2n}$ is called **totally real with respect to $I$** if $V$ is transverse to $JV$. We denote by $\mathcal{T}R(I; n) \subset Gr(2n, n)$ the set of totally real subspaces with respect to $I$. Here $Gr(2n, n)$ is the set of $n$-dimensional subspaces of $\mathbb{R}^{2n}$.

We note that each $\mathcal{T}R(I; n) \subset Gr(2n, n)$ is an open subset of $Gr(2n, n)$ and the union

$$\mathcal{T}R(n) := \bigcup_{I \in J(2n)} \mathcal{T}R(I; n) \to J(2n)$$

forms a fiber bundle over $J(2n)$.

We denote by $D_\epsilon(0)$ the open–closed half-disc with boundary $\partial D_\epsilon(0) = (-\epsilon, \epsilon) \subset \mathbb{R} \subset \mathbb{C}$ and denote by $z = (x, y)$ or $z = x + iy$ the standard coordinates of $\mathbb{R}^2$ or of $\mathbb{C}$.

Let $J : z \mapsto J(z)$ be a $W^{1,p}$ map associating with $z \in D_\epsilon(0)$ a linear complex structure $J(z) \in J(2n)$, and let $z \mapsto C(z)$ be an $L^p$ map with $C(z) \in L^p_r(\mathbb{R}^{2n})$, where $L^p_r(\mathbb{R}^{2n})$ is the set of endomorphisms of $\mathbb{R}^{2n}$. We denote the set of them by

$$W^{1,p}(D_\epsilon(0), J(2n)), \quad L^p(D_\epsilon(0), L^p_r(\mathbb{R}^{2n})), \quad$$

respectively. We assume $p > 2$.

For a given map $J \in W^{1,p}(D_\epsilon(0), J(2n))$, we consider a section $\tau : (-\epsilon, \epsilon) = \partial D_\epsilon(0) \to \mathcal{T}R(J(x); n)$ of $J^*\mathcal{T}R(n)$. We note that, if $J \in W^{1,p}(D_\epsilon(0), J(2n))$ with $p > 2$, then $J|_{(-\epsilon, \epsilon)}$ lies in $W^{1-1/p,p}(\partial D_\epsilon(0), J(2n))$ by the trace theorem (see (Rud73) for example). We require that the map $\tau$ lies in $W^{1-1/p,p}$ as a map from $(-\epsilon, \epsilon)$ into $Gr(2n, n)$ so that its restriction to the boundary becomes a continuous map.

For each given triple $((J; \tau), C)$, we consider the first-order elliptic boundary value problem

$$\begin{cases}
\frac{\partial \xi}{\partial x} + J(z) \frac{\partial \xi}{\partial y}(z) + C(z)\xi(z) = 0, \\
\xi(x, 0) \in \tau(x) \quad \text{for } x \in (-\epsilon, \epsilon).
\end{cases} \quad (8.8.58)$$
Abusing the notation slightly, we denote by the symbol \( i \) either the complex number \( i \) or the corresponding standard complex structure on \( \mathbb{C}^n \cong \mathbb{R}^{2n} \) and denote by \( \overline{\partial} \) the standard Cauchy–Riemann operator \( \partial/\partial x + i \partial/\partial y \). Similarly, we also denote by \( \partial J \) the linear operator \( \xi \mapsto \partial_\xi \partial + J \partial_\xi \).

The main result of this section is the following similarity principle for the boundary-value problem. This result was proved in Theorem 2.1 (Oh97a), which is the boundary analog to that of Floer, Hofer and Salamon for the interior case (FHS95).

**Theorem 8.8.2** (Similarity principle) Let \( (J; \tau, C) \) be as above, let \( p \in (2, \infty) \) and assume \( \xi \in W^{1,p}(D_\epsilon(0), \mathbb{R}^{2n}) \). Then there exists \( \delta \in (0, \epsilon) \) and maps \( \Phi \in W^{1,p}(D_\delta(0), \mathbb{R}^{2n}) \) such that \( \Phi(z) \) is invertible for all \( z \in D_\delta(0) \), and \( \sigma \in C^\infty(D_\delta(0), \mathbb{R}^{2n}) \) such that

\[
\xi(z) = \Phi(z)\sigma(z), \quad J(z)\Phi(z) = \Phi(z)i
\]

and, if we consider \( \sigma \) as a map from \( D_\delta(0) \) to \( \mathbb{C}^n \), it satisfies the classical Riemann–Hilbert problem

\[
\begin{align*}
\frac{\partial \sigma}{\partial z} &= 0, \\
\sigma(x, 0) &\in \mathbb{R}^n \subset \mathbb{C}^n.
\end{align*}
\] (8.8.59)

**Proof** We first find a map \( \Psi \in W^{1,p}(D_\epsilon(0), GL(2n, \mathbb{R})) \) such that

\[
\begin{align*}
J(z)\Psi(z) &= \Psi(z)i \quad \text{in } D_\epsilon(0), \\
\Psi(x, 0)\mathbb{R}^n &= \tau(x) \quad \text{on } \partial D_\epsilon(0).
\end{align*}
\]

Since \( \partial D_\epsilon(0) = (-\epsilon, \epsilon) \) is contractible, we choose a global frame of the real vector bundle \( \tau \rightarrow \partial D_\epsilon(0) \), denoted by \( \{\Psi_1(x, 0), \ldots, \Psi_n(x, 0)\} \), which is complex-linearly independent (with respect to \( J(x) \)) for each \( x \in \partial D_\epsilon(0) \). Therefore, by requiring \( ie_j \) to be mapped to \( J(x, 0)\Psi_j(x, 0) \) for \( j = 1, \ldots, n \) and taking \( \Psi_j \) as its \( j \)th column, we can associate a matrix \( \Psi(x, 0) \) in \( GL(2n, \mathbb{R}) \) that satisfies

\[
J(x, 0)\Psi(x, 0) = \Psi(x, 0)i.
\]

Since \( \tau : (-\epsilon, \epsilon) \rightarrow Gr(2n, n) \) is a map in \( W^{1-1/p, p} \), we can choose \( \Psi_j \) to lie in \( W^{1-1/p, p} \) and hence \( \Psi \in W^{1-1/p, p}(\partial D_\epsilon(0), GL(2n, \mathbb{R})) \).

Now it is easy to extend this map \( x \mapsto \Psi(x, 0) \) to the interior so that

\[
J(z)\Psi(z) = \Psi(z)i \quad \text{in } D_\epsilon(0)
\]

and \( \Psi \) lies in \( W^{1, p}(D_\epsilon(0), GL(2n, \mathbb{R})) \).
Exercise 8.8.3  Prove this claim.

Next we define a map \( \eta \in W^{1,p}(D_\epsilon(0), \mathbb{R}^{2n}) \) by the equation \( \xi(z) = \Psi(z)\eta(z) \). If we regard \( \eta \) as a map to \( \mathbb{C}^n \), then \( \eta \) satisfies the boundary condition

\[
\eta(x, 0) \in \mathbb{R}^n, \quad x \in (-\epsilon, \epsilon),
\]

and the equation

\[
0 = \bar{\partial}_J \xi + C(z)\xi = \frac{\partial \xi}{\partial x} + J(z)\frac{\partial \xi}{\partial y}(z) + C(z)\xi(z).
\]

Here we have

\[
\frac{\partial \xi}{\partial x}(z) = \psi(z)\frac{\partial \eta}{\partial x}(z) + \frac{\partial \psi}{\partial x}(z)\eta(z)
\]

and

\[
J(z)\frac{\partial \xi}{\partial y}(z) = J(z)\Psi(z)\frac{\partial \eta}{\partial y}(z) + J(z)\frac{\partial \Psi}{\partial y}(z)\eta(z)
= \Psi(z)\frac{\partial \eta}{\partial y}(z) + J(z)\frac{\partial \Psi}{\partial y}(z)\eta(z).
\]

By substituting these into the last equation, we show that \( \eta \) satisfies the equation

\[
\begin{cases}
\bar{\partial}\eta + \bar{C}\eta = 0, \\
\eta(x, 0) \in \mathbb{R}^n,
\end{cases}
\]

where \( \bar{C} \) is the multiplication operator of the real matrix-valued function

\[
\bar{C}(z) = \Psi^{-1}(z) \left( \frac{\partial \psi}{\partial x}(z) + J(z)\frac{\partial \Psi}{\partial y}(z) \right) + \Psi^{-1}(z)C(z)\Psi(z).
\]

Since \( C \in L^p \), \( J \in W^{1,p} \) and \( \Psi \in W^{1,p} \), it follows that \( \bar{C} \) again lies in \( L^p \).

But we note that \( \bar{C} \) is not complex linear, i.e., does not commute with complex multiplication by \( i \). We will now find a complex-linear matrix \( B(z) \in GL(2n, \mathbb{R}) \) depending on \( \eta \) regarded as an element lying in \( GL(n, \mathbb{C}) \), such that \( \bar{C}(z)\eta(z) = B(z)\eta(z) \). We decompose \( \bar{C} \in L^p(D_\epsilon(0), GL(2n, \mathbb{R})) \),

\[
\bar{C}(z) = \bar{C}_r(z) + \bar{C}_a(z),
\]

where \( \bar{C}_r(z) \), \( \bar{C}_a(z) \) are the complex linear and anti-linear parts of \( \bar{C}(z) \). Indeed, we have

\[
\bar{C}_r(z) = \frac{1}{2}(\bar{C}(z) - i\bar{C}(z)i), \quad \bar{C}_a(z) = \frac{1}{2}(\bar{C}(z) + i\bar{C}(z)i).
\]

Denote by \( \Gamma : \mathbb{C}^n \to \mathbb{C}^n \) the complex conjugation map which is complex anti-linear. Then we find a complex-linear map \( D(z) \) such that \( D \in L^\infty(D_\epsilon(0), GL(n, \mathbb{C})) \) that satisfies \( D(z)\eta(z) = \overline{\eta(z)} = \Gamma(\eta(z)) \). In fact, we have the explicit formula
$$D(z) = \begin{cases} \langle \eta(z) \rangle \overline{\langle \eta(z) \rangle} / |\eta(z)|^2, & \text{when } \eta(z) \neq 0, \\ 0, & \text{when } \eta(z) = 0, \end{cases}$$

where $|\eta(z)\rangle \overline{\langle \eta(z) \rangle}$ is Dirac’s ‘ket’-operator notation for $(\eta(z)) \overline{\eta(z)}$ for a column vector $\eta(z)$. Since $\eta \in W^{1,p}$ and hence is continuous, the set $\{ z \in D_\epsilon(0) \mid \eta(z) = 0 \}$ is measurable. From the definition, we have $|D(z)| \leq 1$ and hence $D \in L^\infty(D_\epsilon(0), GL(n, \mathbb{C}))$ as required. Then we can check that $\eta$ satisfies

$$\begin{cases} \bar{\partial} \eta + B \eta = 0, \\ \eta(x, 0) \in \mathbb{R}^n, \end{cases}$$

where $B(z) = \bar{C}_\epsilon + \bar{C}_n \bar{D}(z)$. By construction $B$ is complex linear. This enables us to prove the following lemma.

**Lemma 8.8.4** Let $\text{Tr} : W^{1,p}((D_\epsilon(0), \partial D_\epsilon(0)), (\mathbb{C}^n, \mathbb{R}^n)) \to W^{1-1/p,p}(\partial D_\epsilon(0), \mathbb{R}^n)$ be the trace map. Then the linear map

$$(\bar{\partial}, \text{Tr}) : W^{1,p}((D_\epsilon(0), \partial D_\epsilon(0)), (\mathbb{C}^n, \mathbb{R}^n))$$

$$\to L^p(D_\epsilon(0), \mathbb{C}^n) \times W^{1-1/p,p}(\partial D_\epsilon(0), \mathbb{R}^n)$$

is surjective with Fredholm index $n$.

**Exercise 8.8.5** Prove this lemma.

We choose a cut-off function $\chi : \mathbb{H} \to [0, 1]$ such that

$$\chi(z) = \begin{cases} 1 & \text{on } D_{1/2}(0), \\ 0 & \text{outside } D_1(0) \end{cases}$$

and define $\chi_\delta(z) = \chi(z/\delta)$. Then a simple calculation leads to

$$\|\chi_\delta B \eta\|_p \leq C \|\eta\|_{1,p} \|\chi_\delta B\|_p \leq C(\delta) \|\eta\|_{1,p}$$

with constants $C(\delta)$ satisfying $C(\delta) \to 0$ as $\delta \to 0$. Therefore, for a sufficiently small $\delta > 0$, the map

$$(\bar{\partial} + B_\delta, \text{Tr}) : W^{1,p}((D_\epsilon(0), \partial D_\epsilon(0)), (\mathbb{C}^n, \mathbb{R}^n))$$

$$\to L^p(D_\epsilon(0), \mathbb{C}^n) \times W^{1-1/p,p}(\partial D_\epsilon(0), \mathbb{R}^n)$$

for $B_\delta = \chi_\delta B$ is still surjective with index $n$ by the openness of the surjectivity and invariance of the index under continuous deformations. We define

$$\Theta_\delta \in W^{1,p}((D_\epsilon(0), \partial D_\epsilon(0)), (GL(n, \mathbb{C}), GL(n, \mathbb{R})))$$

to be the fundamental solution of

$$\begin{cases} \bar{\partial} \Theta + B_\delta \Theta = 0 & \text{in } D_\delta(0), \\ \Theta(x, 0) = Id \in GL(n, \mathbb{R}) & \text{on } \partial D_\delta(0). \end{cases}$$

(8.8.60)
Such a solution exists and is unique if \(0 < \delta < \epsilon\) is sufficiently small. Now we define \(\sigma : B_\delta(0) \to \mathbb{C}^n\) by the equation \(\eta(z) = \Theta_\delta(z)\sigma(z)\). Then, for \(z \in D_\delta(0)\), a simple computation via (8.8.60) shows that \(\sigma\) satisfies the required equation, which finishes the proof. \(\square\)

Obviously the similarity principle for the interior is subsumed into Theorem 8.8.2. We mention one immediate consequence of the proof thereof on the structure of interior singularities of \(J\)-holomorphic curves.

**Corollary 8.8.6** Let \(J\) be any compatible almost-complex structure of \((M, \omega)\). Consider a non-constant \(J\)-holomorphic map \(u : (\Sigma, j) \to (M, J)\). Let \(du(z_0) = 0\) for \(z_0 \in \Sigma\). Then there exists some \(k \in \mathbb{N}\) such that

\[
j^{k-1}(u)(z_0) = 0 \quad \text{but} \quad j^k u(z_0) \neq 0
\]

and there exists a complex coordinate \(z\) at \(z_0\) such that

\[
u(z) = \bar{v}z^k + O(|z|^{k+1}), \quad \bar{v} \in T_{u(z_0)}M.
\]

In particular, there exists an open neighborhood \(V \subset \Sigma\) of \(z_0\) such that \(du(z) \neq 0\) for all \(z \in V \setminus \{z_0\}\).

Motivated by this corollary, we introduce the following notion.

**Definition 8.8.7** For a non-constant \(J\)-holomorphic map \(u : (\Sigma, j) \to (M, J)\), we define the ramification order of \(u\) at \(z_0\) to be the integer \(k\) mentioned in the above corollary. We call \(j^k u(z_0)\) the principal jet of \(u\).

One consequence of the above corollary is that the principal jet of any \(J\)-holomorphic map is holomorphic in that the jet lies in \(J_{\delta_0}^{(k,0)}(M)\). (We refer the reader to (Oh11a) for a detailed discussion on the structure of ramification profiles and of principal jets of \(J\)-holomorphic maps at critical points.)

The proof of similar statements for the boundary points requires further examination of the boundary behavior, but the above boundary similarity principle will still be useful.

We now state a few immediate consequences of the similarity principle.

**Proposition 8.8.8** Let \(J_0\) be an almost-complex structure on \(\mathbb{R}^{2n}\) and let \(R\) be a totally real submanifold of \(J_0\). Assume the map \(J : (z, p) \mapsto J(z, p)\) is a smooth map from \(D_\epsilon(0) \times M\) to \(\mathcal{J}(2n)\), and suppose

\[
J(z, p) = J_0(p) \quad z = (x, 0) \in \partial D_\epsilon(0) \quad (8.8.61)
\]

for a fixed \((x, 0)\)-independent \(J_0\) on \(\mathbb{R}^{2n}\). Consider a smooth solution
8.8 The similarity principle and the local structure of the image

\[ w : (D_\epsilon(0), \partial D_\epsilon(0)) \to (\mathbb{R}^{2n}, R) \]

of

\[
\begin{aligned}
\frac{\partial w}{\partial x} + J(z, w) \frac{\partial w}{\partial y} &= 0, \\
w(x, 0) \in R, \quad \text{for } x \in (-\epsilon, \epsilon) = \partial D_\epsilon(0).
\end{aligned}
\]

(8.8.62)

Then there exists some \( \delta > 0 \) such that \( dw(z) \neq 0 \) for all \( z \) with \( 0 < |z| < \delta \).

**Proof** If \( dw(0) \neq 0 \), we are done and so assume \( dw(0) = 0 \). By differentiating the given Cauchy–Riemann equation by \( x \), it follows that \( \xi = \partial w / \partial x \) satisfies the equation

\[
\begin{aligned}
\frac{\partial \xi}{\partial x} + J(z, w(z)) \frac{\partial \xi}{\partial y} + \frac{\partial J(z, w(z))}{\partial x} \frac{\partial w}{\partial y} &= 0, \\
\xi(x, 0) \in T_{w(x, 0)} R.
\end{aligned}
\]

But we can write

\[
\frac{\partial J(z, w(z))}{\partial x} \frac{\partial w}{\partial y} = \frac{\partial J(z, w(z))}{\partial x} J(z, w(z)) \xi
\]

since

\[
\frac{\partial w}{\partial y} = J(z, w(z)) \frac{\partial w}{\partial x} = J(z, w(z)) \xi.
\]

If we write

\[
C(z) := \frac{\partial J(z, w(z))}{\partial x} J(z, w(z))
\]

\( \xi \) will satisfy an equation of the type given in Theorem 8.8.2. Therefore the theorem implies that there exists \( \Phi \in C^\infty(D_\delta(0), GL(\mathbb{R}^{2n})) \) such that \( \xi(z) = (\partial w / \partial x)(z) = \Phi(z) \sigma(z) \) with \( \sigma \) holomorphic and \( \sigma(0) = 0 \). And \( (\partial w / \partial x)(z) \neq 0 \) if and only if \( \sigma(z) \neq 0 \). Since \( \sigma \) is holomorphic, from the classical fact on the holomorphic functions it follows that either \( \sigma \equiv 0 \) on a neighborhood of \( 0 \) or there exists some \( \delta > 0 \) such that \( \sigma(z) \neq 0 \) for any \( z \) with \( 0 < |z| < \delta \). This finishes the proof. \( \Box \)

The next theorem is a sort of unique continuation result especially at the boundary point.

**Theorem 8.8.9** Let \( J, R \) as in Proposition 8.8.8. Suppose

\[ w_1, w_2 : (D_\epsilon(0), \partial D_\epsilon(0)) \to (\mathbb{R}^{2n}, R) \]

are smooth and satisfy (8.8.62). If the \( \infty \)-jets of \( w_1, w_2 \) coincide at 0, then \( w_1 \equiv w_2 \).
Local study of J-holomorphic curves

Proof Without any loss of generality, we may assume \( w(0,0) = 0 \in R \subset \mathbb{R}^{2n} \) and \( T_0 R = \mathbb{R}^n \subset \mathbb{C}^n \cong \mathbb{R}^{2n} \). We choose a boundary flattening local diffeomorphism \( \psi : U \to \mathbb{R}^{2n} \) onto its image on an open neighborhood of 0 in \( \mathbb{R}^{2n} \) such that \( \psi(\mathbb{R}^n \cap U) \subset R \), with \( \psi(0) = 0 \). Then (8.8.62) is equivalent to

\[
\frac{\partial \bar{w}}{\partial x} + \psi^* J_{\bar{w}} \frac{\partial \bar{w}}{\partial y} = 0, \quad \bar{w}(x,0) \in \mathbb{R}^n, \quad x \in (-\epsilon, \epsilon),
\]

where \( \bar{w} = \psi^{-1} \circ w, J_{\bar{w}}(w) = J(z,w) \) and \( \psi^* J_{\bar{w}}(w) = T \psi^{-1} J(z,w) T \psi \). Since \( w_1 \) and \( w_2 \) have the same \( \infty \)-jets, \( \bar{w}_1 \) and \( \bar{w}_2 \) have the same \( \infty \)-jets at 0. Now we consider the difference

\[
\xi(z) := \bar{w}_2(z) - \bar{w}_1(z).
\]

Then a straightforward computation shows \( \xi \) satisfies (8.8.62) for

\[
J(z) = \psi^* J_{\bar{w}}(\bar{w}_2(z))
\]

and \( C(z) \) is the linear map determined by

\[
C(z) \xi := \left( \int_0^1 D(\psi^* J_{\bar{w}}(s\bar{w}_1 + (1-s)\bar{w}_2)) ds \right) \cdot \xi \cdot \frac{\partial \bar{w}_1}{\partial y}.
\]

We also have \( j^\infty \xi(0,0) = 0 \). Applying Theorem 8.8.2, we find \( \Phi(z) \in GL(\mathbb{R}^{2n}) \) with \( \xi(z) = \Phi(z) \sigma(z) \) for holomorphic \( \sigma \). Since \( \Phi(0) \) is invertible, we also have \( j^\infty \sigma(0,0) = 0 \). Since \( \sigma \) is holomorphic, this implies \( \sigma \equiv 0 \) and hence \( \xi \equiv 0 \). This proves that \( \bar{w}_1 \equiv \bar{w}_2 \) and hence \( w_1 \equiv w_2 \) near 0.

Now let \( z_0 \in D_\epsilon(0) \) and consider the line segment \( t \in [0,1] \to t z_0 \). We define the subset \( K \subset [0,1] \) by \( K = \{ t \in [0,1] \mid w_1(t z_0) = w_2(t z_0) \} \). We will show that \( K = [0,1] \). \( K \) is clearly a closed subset and the above discussion implies \( K \supset [0,\delta] \) for some \( \delta > 0 \). Let \( 0 < t_0 \leq 1 \)

\[
t_0 = \sup \{ t_1 \mid [0,t_1] \subset K \}.
\]

We claim \( t_0 = 1 \). By definition, we have \( w_1(t z_0) \equiv w_2(t z_0) \) for all \( t \) with \( 0 \leq t \leq t_0 \). Suppose to the contrary that \( t_0 < 1 \). By applying the above discussion at \( t_0 z_0 \in D_\epsilon(0) \), we obtain a holomorphic map \( \sigma \) such that \( \sigma(t z_0) \equiv 0 \) for all \( t \) with \( 0 \leq t \leq t_0 \). This then implies \( \sigma \equiv 0 \) in a small neighborhood \( t_0 z_0 \) and, in particular, \( \sigma(t z_0) \) vanishes for \( t \in (t_0 - \delta, t_0 + \delta) \subset [0,1] \) for a sufficiently small \( \delta > 0 \). This in turn implies \( w_1(t z_0) \equiv w_2(t z_0) \) for \( t \in (t_0 - \delta, t_0 + \delta) \) and hence on \( [0,t_0 + \delta] \), which contradicts the definition of \( t_0 \). This finishes the proof. □

We refer the reader to (Oh97a) for further applications to the study of the structure of boundary singularities and to the continuation of the image of pseudoholomorphic discs along the boundary.
In this chapter, we provide a global study of $J$-holomorphic curves. First we give the definition of basic moduli spaces of $J$-holomorphic curves. Then we give the definition of the moduli space of stable maps introduced by Kontsevich (Kon95) in the $C^\infty$ setting. In particular, we explain the precise definition of stable map topology and prove its compactness and Hausdorffness following the proofs given by Fukaya and Ono in (FOn99).

We will largely restrict ourselves to the closed case, referring readers to (FOOO09), (L02) for a complete discussion of the case of bordered Riemann surfaces. We just quote basic theorems therefrom that we will need for the study of Lagrangian Floer homology in Volume 2 of the present book.

9.1 The moduli space of pseudoholomorphic curves

Gromov’s nonlinear Cauchy–Riemann equation is an equation of “conformal structures of a compact oriented surface $\Sigma$ and a map from $\Sigma$ to an almost Kähler manifold $(M, \omega, J)$”. In this section, we provide the definition of the moduli space of such pairs $(j, u)$.

We first describe the basic objects of study. We fix a closed oriented surface $\Sigma$ of Euler characteristic

$$\chi(\Sigma) = 2 - 2g$$

with $g$ its genus, $2g = \text{rank } H_1(\Sigma, \mathbb{Z})$. We denote by $[\Sigma]$ the fundamental class of $\Sigma$. When we are given a continuous map $u : \Sigma \to M$, it defines the natural element

$$u_*[\Sigma] \in H_2(M, \mathbb{Z})$$

which we denote by $[u]$. We denote by $A$ a general element of $H_2(M, \mathbb{Z})$. When $u$ is differentiable, the integral
\[ \int u^* \omega \]

is defined, and depends only on the homology class \([u] = u_*[\Sigma]\). This is called the \textit{symplectic area} of the differentiable map \(u\).

Now we fix a diffeomorphism type of the surface and denote by \(\Sigma\) any such surface. Then we consider pairs \((j, u)\) of a "conformal structure of \(\Sigma\) and a map". Note that the group \(\text{Diff}(\Sigma)\) acts on the set of pairs by the push-forwards

\[ (\phi, (j, u)) \mapsto (\phi_* j, u \circ \phi^{-1}). \]

We denote

\[ \bar{\partial}_J(j, u) = \frac{du + J \circ du \circ j}{2}, \]

highlighting the dependence of this on the domain complex structure \(j\). We fix a Hermitian metric \(h\) of \(j\) on \(\Sigma\) and any fixed reference compatible metric \(g\) on \(M\).

We would like to study the solution set of the Cauchy–Riemann equation

\[ \bar{\partial}_J(j, u) = 0, \quad \int |du|^2 < \infty \]

for the variable \((j, u)\). Here the energy density \(|du|^2\) is computed in terms of the metrics \(h\) and \(g\).

For a bordered Riemann surface \(\Sigma\) with nonempty boundary \(\partial \Sigma\), we consider maps \(w : \Sigma \to M\) satisfying

\[ \bar{\partial}_J(j, w) = 0, \quad w(\partial \Sigma) \subset L, \quad \int |dw|^2 < \infty \]

for a given Lagrangian submanifold \(L\).

One may replace the Lagrangian submanifold \(L\) by a totally real submanifold insofar as the ellipticity of the boundary-value problem is concerned, as we demonstrated in the previous chapter. The following lemma is the crucial geometric property of Lagrangian submanifolds which distinguishes them from totally real submanifolds.

**Lemma 9.1.1** Let \(L\) be a Lagrangian submanifold of \((M, \omega)\). Suppose that two maps \(w_1, w_2 : (\Sigma, \partial \Sigma) \to (M, L)\) have the same relative homology class \([w_1] = [w_2] \in H_2(M, L)\). Then we have

\[ \int w_1^* \omega = \int w_2^* \omega. \]
9.1 The moduli space of pseudoholomorphic curves

Proof  By the assumption, the chain $w_1 - w_2$ is homologous to a chain $C$ whose support is contained in $L$, i.e., there exists a 3-chain $D$ in $M$ such that

$$\partial D = w_1 - w_2 - C.$$ 

Since $d\omega = 0$ and $\int_C \omega = 0$, Stokes’ formula implies

$$0 = \int_{\partial D} \omega = \int w_1^* \omega - \int w_2^* \omega - \int C \omega = \int w_1^* \omega - \int w_2^* \omega,$$

which finishes the proof. □

Therefore, if one fixes the homology class of the map $u$ as

$$u_*[\Sigma] = A \in H_2(M)$$

or

$$w_*[\Sigma, \partial \Sigma] = B \in H_2(M, L)$$

for the Lagrangian boundary $L$, the symplectic area of the map $u$ (respectively $w$) remains constant. Then, for any $J$-holomorphic map $u$ in either of the two cases, we have the identity

$$\int u^* \omega = E_J(u) = \frac{1}{2} \int |du|^2,$$  \hspace{1cm} (9.1.1)

where we denote by $E_J(u)$ the harmonic energy of $u$ with respect to the metric $g_J = \omega(-, J\cdot)$.

However, in practice, we need to vary the almost complex structure inside $\mathcal{J}_\omega$ or to consider a family of such almost-complex structures depending on the domain variables of $u$. Therefore, in actual geometric estimates, we need to fix a reference metric and to consider $J$ itself as an auxiliary parameter. In this regard the following parameterized version of the a-priori energy estimate is important.

Lemma 9.1.2  Let $K \subset \mathcal{J}_\omega$ be a compact subset and assume that $M$ is compact. Fix any metric $g$ on $M$. Let $(j, u)$ be a $J$-holomorphic map for some $J \in K$ and let $E(u)$ be the harmonic energy of $u$ with respect to the metric $g$. Suppose that $[u] = A$ in $H_2(M)$ (respectively, $[u, \partial u] = B \in H_2(M, L)$). Then there exists a constant $C = C(K)$ such that

$$E(u) \leq C\omega(A)$$

for all $J$-holomorphic $(j, u)$ with $J \in K$ (respectively, $E(u) \leq C\omega(B)$). Furthermore, the constant $C(K)$ varies continuously over $K$ with respect to the $C^0$ topology of $\mathcal{J}_\omega$. 
Proof. We would like to compare $E_J(u)$ and $E(u)$ for $J \in K$. We recall that a metric is a positive definite quadratic form pointwise and there are constants $c(x), C(x) > 0$ such that

$$c(x)g(x) < g_J(x) < C(x)g(x)$$

at each $x \in M$ that are continuous. Denote

$$c_J = \min_{x \in M} c(x), \quad C_J = \max_{x \in M} C(x).$$

Continuity of $c_J$, $C_J$ over $J$ (in $C^\infty$ topology) follows immediately. Then we have

$$c_J|du|^2 \leq |du_J|^2 \leq C_J|du|^2$$

and so $c_J E(u) \leq E_J(u) \leq C_J E(u)$. Since $E_J(u) = \omega(A)$, we obtain $E(u) \leq (1/c_J)\omega(A)$. By setting $C = C(K) = \min_{J \in K} 1/c_J$, we have obtained inequality. Continuity of the constant $C(K)$ over $K$ follows immediately from the expression $C(K) = \min_{J \in K} 1/c_J$. □

We now study the moduli space of solutions

$$\widetilde{M}^E_0(\Sigma, J) = \{(j, u) \mid (\Sigma, j) \rightarrow (M, J), \bar{\partial}(J, \partial)u = 0,$$

$$\int |du_J|^2 \leq E_0, u(\partial \Sigma) \subset R\} \quad (9.1.2)$$

and its convergence property (in $C^\infty$ topology), for a given constant $E_0 > 0$.

Remark 9.1.3 By the above lemma, if we fix a homology class $A$ as

$$u_*[\Sigma] = A \in \begin{cases} H_2(M) & \text{if } \partial \Sigma = \emptyset, \\ H_2(M, L) & \text{if } \partial \Sigma \neq \emptyset, \end{cases}$$

the finite-energy condition will be automatic for the case where either $\partial \Sigma = \emptyset$ or $R$ is a Lagrangian submanifold $L$.

Let $\{(j_\alpha, u_\alpha)\}$ be a sequence in $\widetilde{M}^E_0(\Sigma, J)$. There are obvious sources of failure of convergence:

1. $\max_{z \in \Sigma} |du_\alpha(z)| \rightarrow \infty$;
2. when $g(\Sigma) \neq 0$, $j_\alpha$ may go to infinity of the space of conformal structures on $\Sigma$, i.e., may degenerate;
3. when $g(\Sigma) = 0$, we have the non-compact group $PSL(2, \mathbb{C})$ (for $S^2$) or $PSL(2, \mathbb{R})$ (for $D^2$) of reparameterizations, i.e., if $(j, u) \in \widetilde{M}^E_0(\Sigma, J)$, then $(\phi^* j, u \circ \phi) \in \widetilde{M}^E_0(\Sigma, J)$ for any $\phi \in PSL(2, \mathbb{C})$ or $PSL(2, \mathbb{R})$. 

9.2 Sachs–Uhlenbeck rescaling and bubbling

For the case of $g(\Sigma) = 0$, we eliminate the non-compactness of $\tilde{M}^E_0(\Sigma, J)$ arising from reparameterizations by studying the convergence on the quotient $\tilde{M}^E_0(\Sigma, J) = \tilde{M}^E_0(\Sigma, J)/\sim$, the set of equivalence classes $[u_\alpha]$, with respect to quotient topology. In other words, the convergence of $[u_\alpha]$ means that there exists $\phi_\alpha \in \text{Aut}(\Sigma)$ such that $u_\alpha \circ \phi_\alpha$ converges. A source of difficulty in studying the compactness property of the moduli spaces is that there is no canonical way of choosing the reparameterizations $\phi_\alpha$, which requires a certain amount of ingenuity in the study of convergence according to the given circumstances.

9.2 Sachs–Uhlenbeck rescaling and bubbling

Now we study the convergence property of a sequence $(j_\alpha, u_\alpha)$ of $J$-holomorphic maps in a fixed homology class, in particular with a uniform bound for the harmonic energy. We would like to emphasize that the main coercive $W^{k,p}$ estimates established in Theorem 8.3.5 for $k \geq 2$ rely on the $W^{1,p}$-norm with $p > 2$. Therefore the crucial matter to examine is to establish the $W^{1,p}$-estimate under the finite-energy assumption, which corresponds to the $L^2$-bound for the derivative of the map. In the presence of higher-regularity estimates, this is equivalent to establishing the uniform derivative bound for the given sequence, after extracting a subsequence with suitable reparameterization of the maps. In this point of view derived from the $C^1$-estimate and the Ascoli–Arzelà theorem, Gromov’s compactness theorem (especially for the maps defined on the Riemann surface of genus 0) can be rephrased as the existence of a uniform derivative bound after taking away a finite number of ‘bubbles’: each bubble is either a $J$-holomorphic sphere (or a disc) and eats up non-zero energy not smaller than a uniform positive lower bound. Such a bubble is constructed by first following the celebrated Sacks–Uhlenbeck rescaling argument on the plane $\mathbb{C} = \mathbb{C}P^1 \setminus \{\infty\}$ and then applying the removable singularity theorem given in the previous chapter.

To highlight the Sachs–Uhlenbeck rescaling process, in this section we will mainly focus on the closed Riemann surface $(\Sigma, j)$ with fixed conformal structure $j$ on $\Sigma$. A complete discussion of Gromov compactness in the setting of stable maps will be given in the next section. In this case, we will just simply call a $(j, J)$-holomorphic map $J$-holomorphic. We have already used the most elementary version of the Sachs–Uhlenbeck rescaling argument in the proof of the $\epsilon$-regularity theorem, Theorem 8.4.1.

Assume that either $M$ is compact without a boundary, or that $(M, \omega, J)$ is tame in the sense of Definition 8.7.7 when $M$ is non-compact. We also fix the domain complex structure $j$ and hence the metric $h$ on $\Sigma$. 
Let $u_\alpha : (\Sigma, j) \to (M, J)$ be a sequence of $J$-holomorphic maps satisfying

1. $E(u_\alpha) < C$,
2. there exists a compact subset $K \subset M$ such that Image $u_\alpha \subset K$ for all $\alpha$.

Then one of the following alternatives holds:

1. there exists a subsequence of $u_\alpha$ such that we can choose a sequence of biholomorphisms $\phi_j$ of $\Sigma$ such that
   \[
   \max_{z \in \Sigma} |d(u_\alpha \circ \phi_j)(z)| \leq C_1
   \]
   for some $C_1 > 0$; or
2. for any sequence of biholomorphisms $\phi_\alpha$, the composition $u_\alpha \circ \phi_\alpha$ has no subsequence $u_{\alpha_j} \circ \phi_{\alpha_j}$ such that $\|d(u_{\alpha_j} \circ \phi_{\alpha_j})\|_{C^0}$ is uniformly bounded.

First, we consider alternative (1). We just denote $u_\alpha \circ \phi_\alpha$ by $u_\alpha$ and will assume that there exists $C_1 > 0$, such that $\|du_\alpha\|_{C^0} \leq C_1$ for all $\alpha$. Since the domain $\Sigma$ is compact, this implies $\|du_\alpha\|_p \leq C_2(p, C_1)$ for any $p > 0$. We fix a $p > 2$ as chosen in Theorem 8.3.5. Then, by applying the main estimate, Theorem 8.3.5, to each element of a covering $\{D_\beta\}$ of $\Sigma$, we have bounds for $\|du_\alpha\|_{1,p}$. Then, by the Sobolev embedding $W^{1,p} \hookrightarrow C^\varepsilon$, $0 \leq \varepsilon < 1 - 2/p$, we can extract a subsequence of $u_\alpha$ that converges to a map $u_0 \in C^\varepsilon$ in the $C^\varepsilon$-norm. Then this limit map $u_0$ is a weak solution of $\bar{\partial}_J u = 0$ lying in $C^\varepsilon$. Then the main regularity theorem, Theorem 8.5.5, implies that $u_0$ is in fact smooth and the convergence becomes $C^\infty$-convergence by the boot-strapping argument. In particular, the limit map $u_0$ will have the same homology class as $[u_\alpha] = \alpha \in H_2(M)$. This ends the discussion for alternative (1).

The next theorem then describes what will happen to a general sequence $u_\alpha$ of $J$-holomorphic maps in alternative (2), i.e., for the case that

\[
E(u_\alpha) < C, \quad \max_{z \in S^2} |du_\alpha(z)| \to \infty. \quad (9.2.3)
\]

We recall from Proposition 8.4.8 that the following constant is strictly positive:

\[
A = A(\omega; J) = \inf_u \left\{ \int u^* \omega \mid \bar{\partial}_J u = 0, u : S^2 \to M, u \text{ non-constant} \right\} > 0.
\]

**Theorem 9.2.1** Let $u_\alpha : \Sigma \to M$ be as in (9.2.3). Then there is a subsequence, again denoted by $\{u_\alpha\}$, a finite set of points $\{x_1, \ldots, x_L\} \subset \Sigma$ for some $L \in \mathbb{Z}_+$ and a $J$-holomorphic map $u_0 : \Sigma \to M$ such that the following hold:
(1) $u_\alpha \to u_0$ in compact $C^1$ topology on $\Sigma \setminus \{x_1, \ldots, x_L\}$ and

(2) the energy density $e(u_\alpha) = |du_\alpha|^2$ considered as a measure $e(u_\alpha) d\mu$ ($\mu$ the measure associated to $(\Sigma, h)$) converges to $e(u_0) d\mu$ plus a sum of point measures,

$$e(u_\alpha) \to e(u_0) + \sum_{i=1}^L m_i \delta_{x_i},$$

where each $m_i \geq A(\omega; J) > 0$.

**Proof**  Fix $r_0 > 0$, and consider the sequence $r_m = 2^{-m} r_0$. Here $m \in \mathbb{Z}_+$ and we choose $r_0$ to be smaller than the injective radius of $(\Sigma, h)$.

For each $m \in \mathbb{Z}_+$, we choose a finite covering of $\Sigma$

$$\mathcal{U}_m = \{D_{r_m}(y_i) \mid y_i \in \Sigma\}$$

so that $D_{r_m/2}(y_i)$ also covers $\Sigma$ and each point $x \in \Sigma$ is covered by balls in $\mathcal{U}_m$ at most $\ell$ times, where we can choose $\ell$ independently of $m$, e.g., the choice $\ell = 3$ will suffice since $\Sigma$ is a two-dimensional surface.

From the assumption $\int_{\Sigma} |du_\alpha|^2 < C$, we have

$$\sum_{i=1}^l \int_{D_{r_m}(y_i)} |du_\alpha|^2 \leq l \cdot C.$$ 

Therefore, there are at most $\lceil lC/\epsilon_0 \rceil + 1$ discs with $\int_{D_{r_m}(y_i)} |du_\alpha|^2 \geq \epsilon_0$. (Here, we choose $\epsilon_0$ to be the constant in the main density estimates. See Theorem 8.4.1 or Proposition 8.4.6.) We call such a disc a ‘bad’ disc.

Since $lC/\epsilon_0$ is independent of $\alpha$, and $\Sigma$ is compact, the centers of these ‘bad’ discs form sequences of points in $\Sigma$ as $\alpha \to \infty$. We denote by

$$\{x_{1,m}^\alpha, \ldots, x_{L,m}^\alpha\}$$

the centers of these bad discs for an integer $L \leq \lceil lC/\epsilon_0 \rceil + 1$. After passing to a subsequence of $\alpha$s, we may assume that they converge to

$$\{x_{1,m}, \ldots, x_{L,m}\}$$

as $\alpha \to \infty$. On choosing a subsequence of $m$s, we may also assume that $x_{i,m} \to x_i$ as $m \to \infty$ for all $i = 1, \ldots, L$, which we will assume henceforth.

Except for these bad discs, we have

$$\int_{D_{r_m}(y_i)} |du_\alpha|^2 \leq \epsilon_0$$

on each good disc $D_{r_m}(y_i)$. On these ‘good’ discs, we may apply the density estimate to obtain bounds for $du_\alpha(x)$ on $D_{r_m}(y_i)$. Hence, it reduces to the first
case we considered and, on choosing a subsequence of \( u_\alpha, u_\alpha \) converges on \( D_{r_\alpha/2}(y) \).

We note that, for any given compact subset \( K \subset \Sigma \setminus \{x_1, \ldots, x_L\} \), we will have

\[
K \subset \Sigma - \bigcup_{k=1}^L D_{r/2}(x_{k,m}^\alpha)
\]

for all sufficiently large \( m \) and \( \alpha \). Now, let \( \alpha \to \infty \) for each fixed \( m \) and then let \( K \) exhausts \( \Sigma \setminus \{x_1, \ldots, x_L\} \) as \( m \to \infty \). By taking a diagonal subsequence of \( \alpha = \alpha(m) \) as \( m \to \infty \) we obtain a limit \( u_\alpha \to u_0 \) on \( \Sigma \setminus \{x_1, x_2, \ldots, x_L\} \) in compact \( C^1 \) topology (and so in compact \( C^\infty \) topology). Since \( \int_{\Sigma \setminus \{x_1, \ldots, x_L\}} |du_\alpha|^2 \leq \liminf_{\alpha \to \infty} \int |du_\alpha|^2 \leq E_0 \) and \( u_0 \) satisfies \( \bar{\partial} J u_0 = 0 \) on \( \Sigma \setminus \{x_1, \ldots, x_L\} \), the removable-singularity theorem implies that \( u_0 \) smoothly extends across \( \{x_1, \ldots, x_L\} \) to the whole domain \( \Sigma \). This finishes the proof of statement (1).

Now we proceed to prove statement (2) of the theorem. We have found the points \( x_1, x_2, \ldots, x_L \), where the \( C^1 \)-convergence fails. For any \( \epsilon > 0 \), the numbers \( b_k^\alpha = \sup \{|du_\alpha(x)| : x \in D_{\epsilon}(x_k)\} \)

must be unbounded as \( \alpha \to \infty \). On taking a subsequence, we may assume that \( b_k^\alpha \to \infty \) as \( \alpha \to \infty \) for each \( k = 1, \ldots, L \). We fix \( \epsilon > 0 \), which is less than one half of the injectivity radius, and \( \frac{1}{4} \min_{i,j}\{d(x_i, x_j) | 1 \leq i \neq j \leq L\} \), then \( D_{2\epsilon}(x_i) \) are embedded and pairwise disjoint.

We then set

\[
m_k := \lim_{\epsilon \to 0} \limsup_{\alpha \to \infty} \int_{D_\epsilon(x_k)} |du_\alpha|^2 - |du_0|^2 \ dA.
\]

We need to prove \( m_k \geq A(\omega, J) \) for any \( 1 \leq k \leq L \). For the following discussion, we will fix \( k \) and so abbreviate \( x_k \) by \( x \) and \( b_k^\alpha \) by \( b_\alpha \). We denote by \( \bar{x}_\alpha \) the point in \( D_\epsilon(x) \) such that

\[
|du_\alpha(\bar{x}_\alpha)| = \max_{y \in D_\epsilon(x)} |du_\alpha(y)| = b_\alpha.
\]

**Lemma 9.2.2** Let \( \epsilon > 0 \) be given. Then we have

\[
b_\alpha = |du_\alpha(\bar{x}_\alpha)| = \max_{y \in D_{2\epsilon}(x)} |du_\alpha(y)|
\]

for all sufficiently large \( \alpha \).

**Proof** By virtue of the convergence in compact \( C^1 \) topology of \( u_\alpha \) on \( \Sigma - \bigcup_{k=1}^L D_\epsilon(x_{k,m}^\alpha) \), we have

\[
\max_{y \in \Sigma - \bigcup_{k=1}^L D_\epsilon(x_k)} |du_\alpha(y)| \leq C(\epsilon) < \infty.
\]
In particular, we have
\[ \max_{y \in D_{2e}(x_k) \setminus D_{e}(x_k)} |d\tilde{u}_\alpha(y)| \leq C(\epsilon). \]
Since \( b_\alpha \to \infty \), eventually we have \( b_\alpha > C(\epsilon) \), which finishes the proof. □

Then, \( \bar{x}_\alpha \to x \) as \( \alpha \to \infty \) because of the set-up and in particular \( D_e(\bar{x}_\alpha) \subset D_{2e}(x) \). We fix our coordinate system on \( D_{2e}(x) \subset \Sigma \) and assume that \( D_{2e}(x) \subset \mathbb{C} \). Now re-normalize \( u_\alpha \) to define a map on \( D_{eb_\alpha}(0) \) by
\[ \tilde{u}_\alpha(z) = u_\alpha \left( \bar{x}_\alpha + \frac{z}{b_\alpha} \right). \]
Then we have \( |d\tilde{u}_\alpha(0)| = 1 \). We note that
\[ \bar{x}_\alpha + \frac{z}{b_\alpha} \in D_{2e}(x) \]
for all \( z \in D_{eb_\alpha}(0) \). This and (9.2.4) imply
\[ |d\tilde{u}_\alpha(z)| = \frac{1}{b_\alpha} |d\tilde{u}_\alpha(\bar{x}_\alpha + z/b_\alpha)| \leq \frac{1}{b_\alpha} \max_{y \in D_{2e}(x)} |d\tilde{u}_\alpha(y)| = \frac{1}{b_\alpha} b_\alpha = 1 \]
for all \( z \in D_{eb_\alpha}(0) \). Furthermore, \( D_{eb_\alpha}(0) \) exhausts the whole \( \mathbb{C} \) as \( \alpha \to \infty \) and satisfies
\[ \int_{D_{eb_\alpha}(0)} |d\tilde{u}_\alpha|^2 \leq E_0. \]

Now fix \( R > 0 \) and consider the \( \tilde{u}_\alpha \)'s on \( D_R(0) \). Then we have \( D_R(0) \subset D_{eb_\alpha}(0) \) eventually as \( \alpha \to \infty \). Since \( |d\tilde{u}_\alpha| \leq 1 \) on \( D_R(0) \), there exists a subsequence of \( \tilde{u}_\alpha \), denoted by \( \tilde{u}_{\alpha,R} \), such that \( \tilde{u}_{\alpha,R} \) converges in \( C^1 \) topology on \( D_{R-1}(0) \) as \( \alpha \to \infty \). We denote the corresponding limit by \( u_R : D_{R-1}(0) \to M \). By the \( C^1 \)-convergence of \( \tilde{u}_{\alpha,R} \to u_R \) and \( |d\tilde{u}_{\alpha}(0)| = 1, |du_R(0)| = 1 \).

We may choose \( R = N \in \mathbb{N} \). Let \( N \to \infty \) and choose a diagonal subsequence \( \alpha = \alpha(N) \) so that
\[ \{ \tilde{u}_{\alpha,N+1} \} \subset \{ \tilde{u}_{\alpha,N} \}. \]
Therefore, the diagonal subsequence has the limit \( u_\infty \) defined on \( \mathbb{C} \). Then by the compact \( C^1 \)-convergence, \( u_\infty \) satisfies \( |d\tilde{u}_\infty| = 1, |d\tilde{u}_\infty| \leq 1 \) and \( \tilde{u}_{\alpha,N} \) converges to \( u_\infty \) pointwise, and hence \( |d\tilde{u}_{\alpha,N}|^2 \to |d\tilde{u}_\infty|^2 \) almost everywhere. By Fatou's lemma, we obtain
\[ \int_{\mathbb{C}} |d\tilde{u}_\infty|^2 \leq \liminf_{N} \int_{\mathbb{C}} |d\tilde{u}_{\alpha,N}|^2 \leq E_0. \]
On the other hand, we fix a conformal identification \( \mathbb{C} \simeq S^2 \setminus \{ p \} \) and consider all the above maps and \( u_\infty \) as a map defined on a subset of \( S^2 \setminus \{ p \} \). By the
removable singularity theorem, $u_\infty$ extends to $S^2$. Since $|du_\infty(0)| = 1$, $u_\infty$ is not constant and hence
\[ E(u_\infty) \geq A(\omega, J) \]
by the definition of $A(\omega, J)$. By restoring the index $k = 1, \ldots, L$ and noting that
\[ \int_{D_\epsilon(x_i)} |du_0|^2 \, dA \to 0 \quad \text{as} \quad \epsilon \to 0, \]
we have obtained
\[
E(u_\infty) = \int_C |du_\infty|^2 \leq \liminf_N \int_{D_N} |\tilde{d}u_{\alpha,N}|^2
\leq \lim_{\epsilon \to 0} \limsup_{N \to \infty} \int_{D_\epsilon(x_i)} \left| |d_{\alpha}|^2 - |du_0|^2 \right| dA = m_k
\]
from the construction of $u_\infty$. This finishes the proof of statement (2) on the convergence of the measure $e(u_\alpha)$.

\[ \square \]

**Remark 9.2.3**

1. The above theorem equally applies to the case with a boundary $\partial \Sigma$. The only difference will be the addition of disc bubble maps $w_\infty : (D^2, \partial D^2) \to (M, L)$. (See the proof of the $\epsilon$-regularity theorem in the previous chapter.)

2. The principal component $u_0$ could be a constant map. (Note that a bubble cannot be a constant map by definition.) We illustrate this phenomenon by invoking the case of a disc map. Consider conformal maps as pictured in Figure 9.1.

Here we identify $\mathbb{R} \times [0, 1]$ with $D^2 \setminus \{ \pm 1 \}$ and denote a holomorphic map $u_r$ with $-1 < r < 1$ that sends $(0, 0) \mapsto (r, 0)$ and $(0, 1) \mapsto (1, 0)$. As $r \to 1 - 0$, the derivative of $u_r$ blows up at $(0, 1)$, which produces a disc bubble $u_\infty$ attached at the point $(0, -1) \in \mathbb{C}$ other points, and the principal part $u_0$ becomes a constant map.

3. In the proof of Theorem 9.2.1, we have produced two $J$-holomorphic maps $u_0$ and $u_\infty$. While $u_\infty$ contributes the delta-measure $m_i \delta_{x_i}$, the image of $u_\infty$ need not be attached to the point $u_0(x_i)$. This could happen if there occurs another bubble on the annular region between the principal component $u_0$ and the bubble component $u_\infty$.

As we have seen from the analysis of the failure of compactness under the energy bound in the previous sections, a suitable compactification of the set of $J$-holomorphic curves should involve singular curves.

Gromov introduced a compactification of pseudoholomorphic curves involving ‘cusp-curves’ (Gr85). This result is generally called Gromov’s compactness theorem. A further underpinning of the proof of the compactness theorem was then provided by Pansu (Pa94) and by Parker and Wolfson.
(PW93) for the closed case and by Ye (Ye94) in general. It was Kontsevich who introduced the concept of stable maps, which provided a compactification of the moduli space of pseudoholomorphic maps that is widely used nowadays. In particular, this compactification leads to a topology that is compact and Hausdorff. It also allows a good infinitesimal deformation theory, which is the perfect obstruction theory in algebraic geometry (LiT98) and the Kuranishi structure (FOn99) (or its derivatives) in the smooth context.

9.3 Definition of stable curves

In this section, we closely follow the exposition of Fukaya and Ono in (FOn99). In particular, we will explain the stable map topology defined therein and give detailed proofs of compactness and Hausdorffness closely following their scheme, but with some more clarifications.

9.3.1 Smooth description of stable curves

We first introduce the domain of stable maps for closed curves. Consider the set of compact Riemann surfaces \((\Sigma_v, j_v)\) with a given finite index set \(V\) whose elements we denote by \(v\). We form the finite disjoint union \(\coprod_{v \in V} \Sigma_v\). Next we consider a finite set of unordered pairs, \(e = \{x_e, y_e\}\),

\[
E = \left\{ \{x_e, y_e\} \mid x_e \neq y_e \in \coprod_{v \in V} \Sigma_v \right\}
\]

such that

\[
\{x_{e}, y_{e}\} \cap \{x_{e'}, y_{e'}\} = \emptyset \quad \text{if} \quad e \neq e'.
\] (9.3.5)

Then we define an equivalence relation \(\sim\) on \(\coprod_{v \in V} \Sigma_v\) by \(x \sim x\) for any \(x \in \coprod_{v \in V} \Sigma_v\) and \(x_e \sim y_e\) if \(\{x_e, y_e\} \in E\), and take the quotient space
Σ = \bigsqcup_{v \in V} \Sigma_v / \sim

equipped with the quotient topology. We call the point \([x_e] = [y_e]\) a double point of \(\Sigma\) and denote the set of double points by \(\text{Sing}(\Sigma)\).

To provide a complex structure on \(\Sigma\) as a (singular) complex variety, we give a homeomorphism of a neighborhood of each double point \([x_e] = [y_e]\) onto a local graph \(xy = 0\) in \(\mathbb{C}^2\) near the origin mapping \([x_e]\) to the origin. We then pull back the standard complex structure of the graph

\[
\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}
\]

induced from \(\mathbb{C}^2\). We denote by \(j = j_e\) this complex structure on \(\Sigma\), and call it the glued complex structure of \(j_e\). This leads to a singular complex algebraic curve.

**Definition 9.3.1** A nodal Riemann surface is the pair \((\Sigma, j)\) defined as above. We call each component \((\Sigma_v, j_v) \hookrightarrow (\Sigma, j)\) an irreducible component.

From now on, when there is no danger of confusion, we will just use \(\Sigma\) to represent the pair \((\Sigma, j)\). By construction, \(\Sigma\) has the following properties:

(a) \(\pi_v : \Sigma_v \to \Sigma\) is a continuous map (normalization);
(b) \(\Sigma\) locally looks like the graph of the equation \(xy = 0\) in \(\mathbb{C}^2\) near each double point;
(c) the only allowed singularities of \(\Sigma\) are of the type given in (b).

We note that property (c) follows from the requirement (9.3.5) and property (b).

**Definition 9.3.2** A pre-stable curve is a pair \((\Sigma, z)\), where \(\Sigma = (\Sigma, j)\) is a nodal Riemann surface and \(z\) is a finite set \(z = \{z_1, \ldots, z_m\} \subset \Sigma\) such that

(d) \(z_i \neq z_j\) for \(i \neq j\) and
(e) \(z_j \in \Sigma \setminus \text{Sing}(\Sigma)\).

We call the points in \(z = \{z_1, \ldots, z_m\}\) marked points.

Next we introduce an equivalence relation on the set of pre-stable curves.

**Definition 9.3.3** Let \((\Sigma, z)\) and \((\Sigma', z')\) be two (marked) pre-stable curves. A continuous map \(\phi : \Sigma \rightarrow \Sigma'\) is called an isomorphism if it is a homeomorphism and if restriction of \(\phi\) to each \(\Sigma_v\) lifts to a biholomorphism onto some irreducible component \(\Sigma'_w\) of \(\Sigma'\). We call \(\phi : (\Sigma, z) \rightarrow (\Sigma', z')\) an isomorphism of pre-stable curves, if \(\phi(z_i) = z'_i\) in addition.
A self-isomorphism \( \phi : (\Sigma, z) \to (\Sigma, z) \) is called an automorphism of \((\Sigma, z)\). We denote by \( \text{Aut}(\Sigma, z) \) the set of automorphisms thereof.

**Definition 9.3.4** A pre-stable curve \((\Sigma, z)\) is called *stable* if \( \# \text{Aut}(\Sigma, z) \) is finite and is called unstable otherwise.

Geometrically, stability of \((\Sigma, z)\) is equivalent to the existence of a hyperbolic metric on \( \Sigma \setminus \{z_1, \ldots, z_m\} \).

In the following example, we give a geometric description of the stability criterion. We recall that the uniformization theorem implies that \( \Sigma = \mathbb{C}P^1 \) has a unique conformal structure (or complex structure).

**Example 9.3.5**

1. For \( \Sigma = \mathbb{C}P^1 \), we have \( \dim_{\mathbb{C}} \text{Aut}(\mathbb{C}P^1) = 3 \) because \( \text{Aut}(\mathbb{C}P^1) = \text{PSL}(2, \mathbb{C}) \).
   Moreover, we have
   \[
   \begin{align*}
   \dim_{\mathbb{C}} \text{Aut}(\Sigma, z_1) &= 2, \\
   \dim_{\mathbb{C}} \text{Aut}(\Sigma, z_1, z_2) &= 1, \\
   \dim_{\mathbb{C}} \text{Aut}(\Sigma, z_1, z_2, z_3) &= 0.
   \end{align*}
   \]

2. For \( \Sigma = T^2 = \mathbb{C}/\mathbb{Z} + i\mathbb{Z} \), \( \dim_{\mathbb{C}} \text{Aut}(T^2) = 1 \), hence \( T^2 \) without marked points is unstable. For \( z \in T^2 \), \( \text{Aut}_{\mathbb{C}}(\Sigma, \{z\}) \) is finite (what is it?) and hence \((\Sigma, z)\) is stable.

3. For a Riemann surface \( \Sigma \) with \( \text{genus}(\Sigma) \geq 2 \), \( \dim_{\mathbb{C}}(\text{Aut}(\Sigma)) = 0 \). Otherwise, there would exist a non-zero holomorphic vector field \( X \) obtained from a continuous family of automorphisms. Regarding \( X \) as a section of the tangent bundle and recalling \( c_1(T\Sigma) = \chi(\Sigma) = 2 - 2g \),
   \[
   \#(\sigma_{T\Sigma} \cap \text{graph}(X)) = 2 - 2g
   \]
   with multiplicities counted, where \( \sigma_{T\Sigma} \) is the zero section of the holomorphic line bundle \( T\Sigma \to \Sigma \). Therefore, if \( g \geq 2 \), \( 2 - 2g < 0 \). But this is a contradiction because the local intersection numbers of any non-zero holomorphic section \( X \) of \( T\Sigma \) with the zero section \( \sigma_{T\Sigma} \) are always non-negative.

**Exercise 9.3.6** Prove \( \text{Aut}(\mathbb{C}P^1, z_1, z_2, z_3) = \{\text{id}\} \) and write down all possible fractional linear transformations that preserve \( \{0\} \subset \mathbb{C} \subset \mathbb{C}P^1 \), \( \{0, \infty\} \subset \mathbb{C}P^1 \).

**Definition 9.3.7** We denote by \( \overline{M}_{g,m} \) the set of isomorphism classes of stable curves of \( m \) marked points and of (arithmetic) genus \( g \), i.e., \( g = \ldots \)
rank $H^{(1,0)}(\Sigma, \mathbb{C})$. We denote by $\mathcal{M}_{g,m}$ the subset of $\overline{\mathcal{M}}_{g,m}$ with smooth curves without nodal points.

The space $\overline{\mathcal{M}}_{g,m}$ consists of many strata that can be classified in terms of a certain collection of decorated planar graphs. We now describe this collection of graphs, called the dual graphs.

### 9.3.2 Dual graphs

Consider a planar graph $T$ with finite topology, i.e., with a finite number of edges and vertices. We denote by $V_T$ and $E_T$ the sets of vertices and (unordered) edges respectively. We remark that the two ends $v_1, v_2$ of the edge $e$ may be the same, which corresponds to a self-intersection point of the curve.

**Definition 9.3.8** For $v \in V_T$, we denote by $E(v) = \text{the set of edges incident to } v$, and define the valence of $V$ by

$$\text{val}(v) = \#(E(v)).$$

When the ends of $e$ are the same $v \in V(T)$, we count $v$ twice.

**Definition 9.3.9** We define the dual graph $D$ associated with the triple $(T, (g_v), o)$ such that the following statements hold.

1. $T$ is a planar graph with $\#(V_T), \#(E_T) < \infty$.
2. A function $V_T \to \mathbb{Z}_+; v \mapsto g_v$.
3. A function $o : \{1, \ldots, m\} \to V_T$.

We call $g_v$ the genus of $v$ and $o^{-1}(v)$ the flag of $v$. We denote $\text{flag}(v) := \#(o^{-1}(v))$ and call it the flag order of $v$; and we call $m$ the flag order of $D$ and denote it by $\text{flag}(D)$.

Denote by $V_T^{\text{int}} \subset V_T$ the set of interior vertices. We warn the reader that we regard any vertex without any flag attached as an interior vertex and the end of the flag as an exterior vertex. An exterior edge is one associated with a flag.

We then define a non-negative integer

$$g := \sum_{v \in V_T^{\text{int}}} g_v + b_1(T) \quad \text{with} \quad b_1(T) := \text{rank } H_1(T; \mathbb{Z}) \quad (9.3.6)$$
called the *genus* of the dual graph $D$. By definition, we have

$$\text{flag}(D) = \sum_{v \in V_T^m} \text{flag}(v).$$

Now we would like to introduce the notion of isomorphisms between two dual graphs. The best way to give the definition of isomorphisms is to take a categorical point of view.

For this purpose, we define an *edge-path* from a vertex $v_1$ to $v_2$ in $T$ to be a linear chain of oriented edges starting from $v_1$ to $v_2$. We form a category out of each planar graph $T$ so that objects are vertices and morphisms are edge-paths.

**Exercise 9.3.10** Prove that the above definition makes $T$ become a category.

We denote the corresponding category by $\mathcal{T}$.

**Definition 9.3.11** We say two dual graphs $D$ and $D'$ have the same combinatorial types and denote $D \sim D'$ if there exists an equivalence $\phi$ between the categories $\mathcal{T}$ and $\mathcal{T}'$ such that $\phi$ satisfies

$$g'_{\phi(v)} = g_v \quad \text{for all } v \in V_T,$$

$$\phi \circ o = o'.$$

We call the functor $\phi$ an isomorphism between the dual graphs, and its isomorphism class the combinatorial type of the graph.

For each $v \in V_T$, we define

$$m_v := \text{val}(v) + \text{flag}(v)$$

for any $v \in V_T$.

**Definition 9.3.12** (Stability criterion) A dual graph $D = (T, (g_v), o)$ is called *stable* if $m_v + 2g_v \geq 3$ for all vertices $v \in V_T$.

Now we give the main definition of this section.

**Definition 9.3.13** Define the set $\text{Comb}(g, m)$ to be the set of isomorphism classes of stable dual graphs $D$ with genus $g$ and $m$.

The following combinatorial result is an important ingredient that provides the upper bound for the number of different types of degenerations of pseudoholomorphic maps of a fixed topological type.
Theorem 9.3.14  Suppose that $T$ is stable and connected. Then we have the inequality

$$
\#(V_T) \leq 2m + 5g - 2, \quad \#(E_T) \leq 2m + 6g - 3
$$

and in particular there are only finitely many combinatorial types of $D$ with fixed $g$ and $m$, i.e., $\#\text{Comb}(g,m) < \infty$.

Proof  First we note that

$$
\#(E_T) = \frac{1}{2} \sum_{v \in V_T} \text{val}(v).
$$

Then the connectedness of $T$ and Euler’s identity give rise to

$$
\chi(T) = 1 - b_1(T) = \#V_T - \#E_T = \frac{1}{2} \sum_{v \in V_T} (2 - \text{val}(v)), \quad (9.3.7)
$$

where $\chi(T)$ is the Euler characteristic of $T$. Next we have

$$
\sum_v g_v \leq g, \quad b_1(T) \leq g. \quad (9.3.8)
$$

Obviously, $\#\{v \in V_T \mid g_v \geq 1\} \leq g$, and

$$
\#\{v \in V_T \mid \exists e = \{v,v\}\} \leq g,
$$

since such $e$ contributes to $b_1(T)$, and hence to $g$. We also have

$$
\#\{v \in V_T \mid \text{flag}(v) \geq 1\} \leq m.
$$

All other vertices satisfy $g_v = 0$ and $\text{flag}(v) = 0$, so the stability condition implies

$$
3 \leq 2g_v + m_v = 2g_v + \text{val}(v) + \text{flag}(v) \Rightarrow \text{val}(v) \geq 3
$$

for such vertices. Altogether, we have proved

$$
\#(V_T) \leq m + 2g + \#\{v \in V_T \mid \text{val}(v) \geq 3\}. \quad (9.3.9)
$$

It remains to estimate $\#\{v \mid \text{val}(v) \geq 3\}$. From (9.3.7), we obtain

$$
\chi(T) \leq \frac{1}{2} \#\{v \in V_T \mid \text{val}(v) = 1\} - \frac{1}{2} \#\{v \in V_T \mid \text{val}(v) \geq 3\}.
$$

Therefore we have

$$
2\chi(T) + \#\{v \mid \text{val}(v) \geq 3\} \leq \{v \mid \text{val}(v) = 1\}. \quad (9.3.10)
$$

But the stability condition implies

$$
\#\{v \mid \text{val}(v) = 1\} \leq \#\{v \mid g_v > 0\} + \#\{v \mid \text{flag}(v) > 1\} \quad (9.3.11)
$$
and we have the upper bound
\[
\#\{v \mid g_v > 0\} \leq g, \quad \#\{v \mid \text{flag}(v) > 1\} \leq m. \tag{9.3.12}
\]
Therefore, on combining (9.3.8), (9.3.10), (9.3.11) and (9.3.12), we have obtained
\[
\#\{v \mid \text{val}(v) \geq 3\} \leq m + g - 2\chi(T) = m + g - 2(1 - b_1(T)) \leq m + 3g - 2.
\]
On substituting this into (9.3.9), we have obtained
\[
\#(V_T) \leq 2m + 5g - 2.
\]
We derive
\[
\#(E_T) = \#(V_T) - (1 - b_1(T)) \leq 2m + 6g - 3.
\]
The finiteness statement then follows because there will be at most
\[
\prod_{m}^{\#(V_T)} = \binom{\#(V_T) + m - 1}{m}
\]
possible choices of the flag \( o \): Altogether, we have an upper bound
\[
\#(\text{Comb}(g, m)) \leq N(g, m)
\]
for some integer \( N(g, m) \) depending only on \( g, m \). This finishes the proof. \( \square \)

**Exercise 9.3.15** Estimate the above integer \( N(g, m) \) in terms of \( g, m \).

Next we introduce a partial order on \( \text{Comb}(g, m) \). Denote by \( D := [T, (g_v), o] \) an isomorphism class represented by the dual graph \( (T, (g_v), o) \). Slightly abusing the notation, we occasionally just denote \( D = (T, (g_v), o) \) when there is no danger of confusion.

**Definition 9.3.16** Let \( D \) and \( \widetilde{D} \) be two elements from \( \text{Comb}(g, m) \). We define \( D > \widetilde{D} \) if \( \widetilde{D} = (\widetilde{T}, (\widetilde{g_v}), \widetilde{o}) \) is obtained from \( D := (T, (g_v), o) \) by a finite chain of operations of replacing a vertex \( v \) of \( T \) by a graph \( T_v \in \text{Comb}(g_v, m_v) \) and leaving the rest of the graph unchanged. See Figure 9.2.

Now we associate with each stable curve \( (\Sigma, z) \) one of the above decorated graphs \( D \) in the following fashion (see Figure 9.3).

- Each vertex \( v \) of \( T_\Sigma \) corresponds to a component \( \{\Sigma_v\} \) of \( \Sigma \).
- Each edge corresponds to a double point of \( \Sigma \) and each flag to a marked point.
- Each vertex is decorated by \( g_v \), the geometric genus of \( \Sigma_v \).
Then the integer
\[ g := \sum_v g_v + \text{rank } H_1(T_\Sigma, \mathbb{Z}) \]
corresponds to the arithmetic genus of the associated stable curve \( \Sigma \) (i.e., \( g = \dim H^1(\Sigma, O) \), where \( O \) is the sheaf of holomorphic functions on \( \Sigma \)). We define \( D_\Sigma = [T_\Sigma, (g_v), o_\Sigma] \) and call \( D_\Sigma \) or \( (T_\Sigma, (g_v), o_\Sigma) \) the dual graph of \( (\Sigma, z) \).

With this correspondence, we have the correspondence
\[ \text{val}(v) \longleftrightarrow \text{sing}(v), \quad \text{flag}(v) \longleftrightarrow \text{mark}(v) \]
if we define \( \text{sing}(v) \) as the number of double points of \( \Sigma_v \) and \( \text{mark}(v) \) as that of marked points on \( \Sigma_v \).

Conversely, we can associate a stable curve \( (\Sigma, z) \) with each stable dual graph.

**Definition 9.3.17** Denote by \( \mathcal{M}_{g,m}^D \) the set of isomorphism classes of stable curves of their dual graphs given by \( D \), and form the union
\[ \overline{\mathcal{M}}_{g,m} := \bigcup_{D \in \text{Com}(g,m)} \mathcal{M}_{g,m}^D. \]
**Exercise 9.3.18** Prove that $\mathcal{M}_{g,m}$ is second countable and locally compact with respect to the natural induced topology on the equivalence classes of stable curves.

Then we can express $\mathcal{M}_{g,m}$ as the countable union

$$\mathcal{M}_{g,m} = \bigcup_{\ell=1}^{\infty} \mathcal{U}_\ell, \quad \mathcal{U}_\ell \subset \overline{\mathcal{U}}_\ell \subset \mathcal{U}_{\ell+1}$$

and each $\overline{\mathcal{U}}_\ell$ is compact. We call any proper sequence $\ell \to [\Sigma_\ell, j_\ell]$ in $\mathcal{M}_{g,m}$ a degeneration sequence.

We leave the proof of the following proposition as an exercise.

**Proposition 9.3.19** Suppose that $[\Sigma_\ell, j_\ell] \in \mathcal{M}_{g,m}$ is a degenerating sequence. Then we have two kinds of degeneration:

(i) collision of two special points and

(ii) degeneration of conformal structures.

**Exercise 9.3.20** Prove this proposition (or see (DM69) for the details). (Hint. Prove its contrapositive. First represent a conformal structure of the associated punctured Riemann surface by its Poincaré metric (the unique hyperbolic metric), and prove that the injectivity radius is uniformly bounded away from zero if degeneration of conformal structures does not occur. Note that the stability implies that the associated Poincaré metric is hyperbolic.)

More geometrically, in terms of the Poincaré metric on $\Sigma \setminus \{z_1, \ldots, z_m\}$, a degeneration of marked Riemann surfaces corresponds to pinching off a short geodesic of the Poincaré metric. Therefore a homologically trivial loop cannot be pinched off unless the disc enclosed by the resulting curve contains at least two marked points in its interior. This is because otherwise such a domain enclosed by a short geodesic cannot carry a metric of constant curvature $-1$.

Now we would like to provide a topology on the set $\overline{\mathcal{M}}_{g,m}$ of marked stable curves, which will reflect these degenerations of marked Riemann surfaces. More precisely, we will provide the topology such that

1. its subspace topology on $\mathcal{M}_{g,m} \subset \overline{\mathcal{M}}_{g,m}$ is equivalent to the original quotient topology $\mathcal{M}_{g,m} = \overline{\mathcal{M}}_{g,m} / \sim$ and

2. each degeneration sequence leads to going down to a dual graph in the partial order defined in Definition 9.3.16.
The basic requirements for such a topology on $\overline{M}_{g,m}$ are the following:

1. It is compact and Hausdorff;
2. If we denote by $\overline{M}_{g,m}^D$ the closure of $M_{g,m}^D$ in $\overline{M}_{g,m}$, then we have
   \[ \overline{M}_{g,m}^D = \bigcup_{D' \leq D} M_{g,m}^{D'}; \]
3. It has a chart such that the coordinate neighborhood is homeomorphic to an orbifold $V/\Gamma$, where $V \subset \mathbb{R}^N$ is a subspace and $\Gamma$ is a finite subgroup of $GL(\mathbb{R}^N)$ (see Figure 9.4);
4. There exists a universal bundle $\mathcal{U}_{g,m} \to \overline{M}_{g,m}$ whose fiber at $[\Sigma, z] \in \overline{M}_{g,m}$ is isomorphic to the stable curve itself $(\Sigma, z)$.

In fact we have the natural forgetful map which forms a locally trivial fibration (in the orbifold sense)

\[ \text{\texttt{forget}}_i : M_{g,m+1} \to M_{g,m}, \]
\[ [j, (z_1, \ldots, z_{m+1})] \to [j, (z_1, \ldots, \hat{z}_i, \ldots, z_{m+1})]. \]

(9.3.13)

Its fiber $\text{\texttt{forget}}_i^{-1}([j, (z_1, \ldots, z_m)])$ can be identified with

\[ \{(j', (z'_1, \ldots, z', \ldots, z'_m)) | z \in \Sigma \setminus \{z'_1, \ldots, z'_m\} \} / \text{Aut}(\Sigma, z'), \]

which is isomorphic to

\[ \{z \in \Sigma \setminus \{z_1, \ldots, z_m\} / \text{Aut}(j, (z_1, \ldots, z_m)). \]
9.4 Deformations of stable curves

This forgetful map continuously extends to the universal fibration \( \Upsilon_{g,m} \rightarrow \mathcal{M}_{g,m} \), which itself is represented by the forgetful map

\[
\text{forget}_{m+1} : \mathcal{M}_{g,m+1} \rightarrow \mathcal{M}_{g,m}.
\]  

(9.3.14)

**Remark 9.3.21** Unlike (9.3.13), the definition of the forgetful map (9.3.14) requires some explanation. After we have forgotten the last \((m + 1)\)th marked point, the irreducible component containing the marked point can become unstable. This can happen only when the component has genus zero and has exactly three special points. There are two cases to consider. One is the case where the three special points consist of two nodal points and one marked point and the other is when they consist of one nodal point and one marked point. For the first case, we just contract the unstable component. One can easily see that all other components are not affected by this process and hence the resulting pre-stable curve is stable. For the second case, we contract the component and leave one remaining marked point on the neighboring component at which the contracted component is rooted. Again the resulting pre-stable curve is stable. Moreover, the total number of marked points is unchanged and the genus remains the same, since we contracted a genus-0 component. This is what we mean by the forgetful map (9.3.14).

We also have the following general lemma whose proof we leave as an exercise

**Lemma 9.3.22**

\[
\text{forget}_{m+n;m} \circ \text{forget}_{m+n';m+n} = \text{forget}_{m+n+n';m+n'} = \text{forget}_{m+n';m} \circ \text{forget}_{m+n+n';m+n'}
\]

in \( \mathcal{M}_{g,m+n+n'} \).

**Exercise 9.3.23** Prove this lemma.

**9.4 Deformations of stable curves**

We first describe the infinitesimal deformation space of \( \mathcal{J}_\Sigma \), the set of complex structures \( j \) on \( \Sigma \), and provide a coordinate atlas for the Deligne–Mumford space \( \mathcal{M}_{g,m} \). Then we will provide the atlas of \( \overline{\mathcal{M}}_{g,m} \) inductively, starting from the lowest-dimensional stratum. Each element in the lowest-dimensional
stratum is the union of surfaces of genus 0 with three special points consisting of either marked points or singular points.

Recall that, on a smooth surface \( \Sigma \), any almost-complex structure has zero torsion and hence is integrable. Therefore we can identify a complex structure with the associated almost-complex structure, and describe the space \( \overline{M}_{g,m} \) in the differential geometric context. We refer readers to (RS06) for an exposition in a similar spirit.

### 9.4.1 Deformations of complex structures

An almost-complex structure \( j : T\Sigma \to T\Sigma \) is an endomorphism satisfying \( j^2 = -id \) on \( T\Sigma \) at each point \( x \in \Sigma \). Denote by \( \mathcal{J}_\Sigma \) the set of almost-complex structures.

The group \( \text{Diff}(\Sigma) \) acts on \( \mathcal{J}_\Sigma \) by the push-forward

\[
(\phi, j) \in \text{Diff}(\Sigma) \times \mathcal{J}_\Sigma \mapsto \phi_* j \in \mathcal{J}_\Sigma,
\]

where \( \phi_* j = d\phi \circ j \circ d\phi^{-1} \). We denote

\[
\mathcal{M}_g := \mathcal{J}_\Sigma / \text{Diff}(\Sigma).
\]

We have the decomposition of the complexified tangent space \( T_\Sigma \mathbb{C} = T\Sigma \otimes_{\mathbb{R}} \mathbb{C} \)

\[
T_\Sigma \mathbb{C} = T^{(1,0)}\Sigma \oplus T^{(0,1)}\Sigma
\]

satisfying the relation

\[
T^{(0,1)}\Sigma = \overline{T^{(1,0)}\Sigma}.
\]

Conversely, for any given complex subspace \( H \subset T_\Sigma \mathbb{C} \) such that

\[
T_\Sigma \mathbb{C} = H \oplus \overline{H},
\]

we can associate an almost complex structure \( j \) on \( T\Sigma \) whose eigenspaces of \( j : T_\Sigma \mathbb{C} \to T_\Sigma \mathbb{C} \) are precisely \( H \) and \( \overline{H} \). Note that we can embed the real tangent bundle \( T\Sigma \) as the fixed-point set of the conjugation on \( T_\Sigma \mathbb{C} \), i.e.,

\[
T\Sigma \cong \left\{ \left. \frac{\xi + \overline{\xi}}{2} \in T_\Sigma \mathbb{C} \right| \xi \in H \right\}.
\]

Under this identification, we define an endomorphism \( j_H : T\Sigma \to T\Sigma \) by the map

\[
j_H \left( \frac{\xi + \overline{\xi}}{2} \right) := \frac{i\xi + i\overline{\xi}}{2}.
\]

By definition, \( j_H^2 = -id \) and so \( j_H \) defines an almost-complex structure on \( \Sigma \). Therefore, a splitting (9.4.15) of the complexified tangent space \( T_\Sigma \mathbb{C} \) is
equivalent to an almost-complex structure of $\Sigma$. We denote by $\Pi : T_C \Sigma \rightarrow T_C \Sigma$ the idempotent associated with $j$ such that its image is $T^{(1,0)} \Sigma$ and its kernel $T^{(0,1)} \Sigma$.

We now study deformations of complex structures on $\Sigma$ by considering the deformations of the idempotent $\Pi$. Let $j$ be a complex structure and let $T_C \Sigma = T^{(1,0)} \Sigma \oplus T^{(0,1)} \Sigma$ be given. The $H$ corresponding to nearby complex structure $j'$ is determined by

$$\overline{H} = \text{Graph } B,$$

where $B \in \Gamma(\text{Hom}_C(T^{(0,1)} \Sigma, T^{(1,0)} \Sigma)) = \Omega^{(0,1)}(T^{(1,0)} \Sigma)$ with $B = 0$ when $j' = j$. Conversely, for any such $B$ with its norm $|B|$ sufficiently small, it determines $j'$ as above. Therefore the space of complex structures on $\Sigma$ is an infinite-dimensional manifold modeled over the affine space

$$\Omega^{(0,1)}(T^{(1,0)} \Sigma).$$

(It is common to denote this just by $\Omega^{(0,1)}(T \Sigma)$ in complex geometry after making the identification of $T^{(1,0)} \Sigma$ with $T \Sigma$, but we do not follow that convention, in order to avoid possible confusion, preferring the explicit description in terms of $T^{(1,0)} \Sigma$.)

We can write the exterior differential $d : \Omega^*(\Sigma, \mathbb{C}) \rightarrow \Omega^{*+1}(\Sigma, \mathbb{C})$, $d = \partial + \overline{\partial}$, where, for $* = 0$, we have

$$\partial = \pi^{(1,0)} \circ d = \frac{d - \sqrt{-1}jd}{2},$$

$$\overline{\partial} = \pi^{(0,1)} \circ d = \frac{d + \sqrt{-1}jd}{2}.$$  

From $d^2 = 0$, we obtain

$$\partial \overline{\partial} + (\partial \overline{\partial} + \overline{\partial} \partial) + \overline{\partial} \overline{\partial} = 0.$$  

By comparing the types of the resulting images, we see that this implies

$$\partial \circ \partial = \overline{\partial} \circ \overline{\partial} = (\partial \circ \overline{\partial} + \overline{\partial} \circ \partial) = 0.$$  

In particular, we have the natural first-order differential operator

$$\overline{\partial} : C^\infty(\Sigma, \mathbb{C}) \rightarrow \Omega^{(0,1)}(\Sigma, \mathbb{C}),$$

(9.4.16)

called the standard Dolbeault operator.

We denote the set of $T^{(1,0)} \Sigma$-valued 0- and $(0, 1)$-forms on $\Sigma$ by

$$\Omega^0(T^{(1,0)} \Sigma) = \Gamma(\Sigma, T^{(1,0)} \Sigma),$$

$$\Omega^{(0,1)}(T^{(1,0)} \Sigma) = \Gamma(\Sigma, \Lambda^{(0,1)} \Sigma \otimes T^{(1,0)} \Sigma),$$

for $\Lambda^{(0,1)} \Sigma$ the $(0, 1)$-forms on $\Sigma$.
respectively. Then we have a natural differential operator
\[ \bar{\partial} : \Omega^0(T^{(1,0)}\Sigma) \to \Omega^{(0,1)}(T^{(1,0)}\Sigma) \] (9.4.17)
that extends the Cauchy–Riemann operator \( \bar{\partial} \) above so that
\[ \bar{\partial}(f\xi) = \bar{\partial}f \otimes \xi + f \bar{\partial}\xi. \] (9.4.18)
When \( \xi = f \partial/\partial z \) in the complex coordinate \( z \) on \( \Sigma \), we have the local expression
\[ \bar{\partial}\xi = \frac{\partial f}{\partial z} \, dz \otimes \frac{\partial}{\partial z}. \]
We compute its associated symbol map now. (See (10.1.2) for the definition of the symbol of a general differential operator.)

**Proposition 9.4.1** The symbol map \( \sigma_1(\bar{\partial}) \) associated with (9.4.18)
\[ \sigma_1(\bar{\partial})(\alpha) : T_z^{(1,0)}\Sigma \to \Lambda_z^{(0,1)}\Sigma \otimes T_z^{(1,0)}\Sigma \] (9.4.19)
for \( 0 \neq \alpha \in T_z^*\Sigma \) is given by the maps
\[ \sigma_1(\bar{\partial})(\alpha)(\xi) = i\alpha^{(0,1)} \otimes \xi \]
at each \( \alpha \in T^*\Sigma \setminus \{0\} \), where \( \alpha = \alpha^{(1,0)} + \alpha^{(0,1)} \). In particular, it is an isomorphism at any \( \alpha \neq 0 \).

**Proof** The proof is left as an exercise. \( \square \)

The following general theorem is a useful ingredient in the study of deformation of complex structures. This theorem holds for general complex vector bundles \( E \) on a complex manifold \( X \). We restrict the discussion to a line bundle on a Riemann surface.

**Theorem 9.4.2** Equipping a holomorphic structure on a complex line bundle \( L \to \Sigma \) is equivalent to having a \( \mathbb{C} \)-linear first-order differential operator \( D : \Omega^0(\Sigma; L) \to \Omega^{(0,1)}(\Sigma; L) \) satisfying
\[ D(f\alpha) = \bar{\partial}f \otimes \alpha + f \, D\alpha, \] (9.4.20)
for all \( f \in C^\infty(\Sigma; \mathbb{C}) \) and \( \alpha \in \Omega^0(\Sigma; L) \). Furthermore for any two such operators \( D \) and \( D' \), there exists a section \( \varphi \in \Omega^{(0,1)}(\Sigma; \mathbb{C}) \) such that \( D' = D + \varphi \), where \( \varphi \) acts as a multiplication operator on \( \Omega^0(\Sigma; L) \) induced by the bundle map
\[ T^{(0,1)}\Sigma \otimes L \to L \]
over \( \mathbb{C} \), i.e., an element in \( \Omega^{(0,1)}(\text{End}(L)) \cong \Omega^{(0,1)}(\Sigma; \mathbb{C}) \).
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Proof  We need only prove that any such operator $D$ produces a holomorphic structure on $L$. For this purpose, we need only provide a holomorphic atlas.

We start by proving the second statement. Since both $D$ and $D'$ satisfy (9.4.20), we have

$$(D' - D)(f\alpha) = f(D' - D)\alpha,$$

which proves that $D' - D$ is a differential operator of order zero, i.e., a section of

$$\text{Hom}_C(\mathcal{L}, \Lambda^{(0,1)}\Sigma \otimes \mathcal{L}) = \text{Hom}_C(T^{(0,1)}\Sigma \otimes \mathcal{L}, \mathcal{L}).$$

Denoting this by $\varphi$, we obtain $D' = D + \varphi$.

We next define local frames of $L$ whose transition functions become (local) holomorphic functions on $\Sigma$. For this, we consider the local sections $\xi$ of $L$ with $D\xi = 0$. We call them local $D$-sections. We claim that, if $\xi_\alpha$ and $\xi_\beta$ are nowhere-vanishing $D$-sections defined on $U_\alpha$ and $U_\beta$, respectively, then there exists a holomorphic function $f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}$ such that $\xi_\beta(z) = f_{\alpha\beta}(z)\xi_\alpha(z)$ for all $z \in U_\alpha \cap U_\beta$. Since $L$ has rank 1, we can certainly express $\xi_\beta(z) = f_{\alpha\beta}(z)\xi_\alpha(z)$ for some complex-valued function $f_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}$. It remains to prove $\overline{\partial}f_{\alpha\beta} \equiv 0$ on $U_\alpha \cap U_\beta$. But we have

$$0 = D\xi_\beta = D(f_{\alpha\beta}\xi_\alpha) = \overline{\partial}f_{\alpha\beta}\xi_\alpha + f_{\alpha\beta}D\xi_\alpha = \overline{\partial}f_{\alpha\beta}\xi_\alpha$$

on $U_\alpha \cap U_\beta$. Since $\xi_\alpha$ does not vanish by assumption, we obtain $\overline{\partial}f_{\alpha\beta} = 0$.

Therefore it remains to prove that at any given point $z_0 \in \Sigma$ there exists an open neighborhood $U_0$ of $z_0$ such that the equation $D\xi = 0$ has a nowhere-vanishing solution on $U$. We choose a contractible coordinate neighborhood $V_0$ of $z_0$ and denote by $z$ the local complex coordinate at $z_0$. Trivializing $L$ on $V_0$, we may regard $L|_{U_0} \cong U_0 \times \mathbb{C}$ and $U_0 \subset \mathbb{C}$. Then, from the first part of the proof, we can write $D = \overline{\partial} + \varphi$ for some $\mathbb{C}$-valued $(0,1)$-form $\varphi$. Now the equation $D\xi$ is reduced to solving the perturbed $\overline{\partial}$-equation $\overline{\partial}\xi + \varphi\xi = 0$ for a complex-valued function $\xi$. Writing

$$\overline{\partial} = \left(\frac{\partial}{\partial \overline{z}} + b\right)d\overline{z}, \quad \xi = a\frac{\partial}{\partial \overline{z}}$$

the equation $\overline{\partial}\xi + \varphi\xi = 0$ is reduced to

$$\frac{\partial a}{\partial \overline{z}} + ba = 0$$

on $U_0$. This can be easily solved in a small neighborhood of $z_0$ by integration with an initial condition $a(z_0) = 1$. By shrinking $U_0$ if necessary, we have constructed a nowhere-vanishing local section $\xi$ near $z_0$. This finishes the proof of the theorem.  \(\square\)
Applying this theorem to the complex line bundle $\mathcal{L} = T(1,0)\Sigma$, we can study deformation of complex structures by studying the deformation of operators $D = \bar{\partial} + \varphi$ over $\varphi$ modulo the equivalence relation induced by the action of $\text{Diff}(\Sigma)$ on $j$.

In summary, we have proved the following lemma.

**Lemma 9.4.3** Let $j \in \mathcal{J}_\Sigma$. Then we have an isomorphism

$$T_j \mathcal{J}_\Sigma \cong \Omega^{0,1}(\Sigma).$$

Next we study the effect on $\varphi$ under the action of $\text{Diff}(\Sigma)$ on $j$, where $\varphi$ is the function appearing in the operator $D = \bar{\partial} + \varphi$ associated with the push-forward $\phi_* j$. Recall the (left) action of $\text{Diff}_0(\Sigma)$ on $j$ given by

$$\phi_* j = d\phi \circ j \circ d\phi^{-1},$$

where $d\phi : T_C \Sigma \to T_C \Sigma$ is the complexified derivative of $\phi$.

Let $\phi_s$ be a smooth one-parameter family of diffeomorphisms on $\Sigma$ for $-\epsilon < s < \epsilon$ with $\phi_0 = \text{id}$ and $d/ds|_{s=0}\phi_s = X$. The infinitesimal action of $X$ on $j$ is given by the Lie derivative

$$\left. \frac{d}{ds} \right|_{s=0} (\phi_s)_* j = -\mathcal{L}_X j.$$

The following proposition connects the study of infinitesimal deformations of complex structures with the Dolbeault cohomology of $\Sigma$.

**Proposition 9.4.4** Let $X \in \Gamma(T\Sigma)$. The Lie derivative $\mathcal{L}_X j$ satisfies $(\mathcal{L}_X j) j + j(\mathcal{L}_X j) = 0$ and so defines a section of $\Lambda_j^{0,1}(T\Sigma)$. Furthermore, we have

$$\Pi \circ \mathcal{L}_X j|_{\mathcal{T}^{0,1}\Sigma} = i \partial \bar{\partial} \xi$$

in complexification, where $\xi = \pi^{(1,0)} X$.

**Proof** In local complex coordinates $z$, the complexification of $j$ can be expressed as

$$j = i \, dz \otimes \frac{\partial}{\partial z} - i \, d\bar{z} \otimes \frac{\partial}{\partial \bar{z}}$$

since $j^2 = -\text{id}$ and the eigenspace with eigenvalue $i$ is given by $\text{span}_\mathbb{C} \{\partial/\partial z\}$ and the one with eigenvalue $-i$ is $\text{span}_\mathbb{C} \{\partial/\partial \bar{z}\}$. Since $X$ is a real vector, we can also write it as $X = \xi + \bar{\xi}$ for the $(1,0)$-vector field $\xi = \pi^{(1,0)} X$, which we can write as $\xi = f(z) \partial/\partial z$ for some local complex-valued function $f$. 

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Lemma 9.4.5

\[ L_X j = \frac{i \partial f}{\partial \bar{z}} \, d\bar{z} \otimes \frac{\partial}{\partial z} - i \frac{\partial f}{\partial z} \, dz \otimes \frac{\partial}{\partial \bar{z}}. \]  

(9.4.21)

Proof  We compute

\[ L_{f(z) \partial/\partial z} \frac{\partial}{\partial z} = - \frac{\partial f}{\partial z} \frac{\partial}{\partial z}, \quad L_{f(z) \partial/\partial z} \frac{\partial}{\partial \bar{z}} = - \frac{\partial f}{\partial \bar{z}} \frac{\partial}{\partial \bar{z}}, \]

from which we also obtain

\[ L_{f(z) \partial/\partial z} (dz) = \frac{\partial f}{\partial z} \, dz, \quad L_{f(z) \partial/\partial z} (d\bar{z}) = 0. \]

Therefore we derive

\[ L_{f(j) \partial/\partial \bar{z}} = i \frac{\partial f}{\partial \bar{z}} \, d\bar{z} \otimes \frac{\partial}{\partial z}. \]

A similar computation also leads to

\[ L_{f(j) \partial/\partial \bar{z}} = - i \frac{\partial f}{\partial \bar{z}} \, dz \otimes \frac{\partial}{\partial \bar{z}}. \]

By adding the two equations, we obtain the formula. \qed

From this lemma, we get

\[ \Pi \circ L_j \big|_{T(1,0)_\Sigma} = i \frac{\partial f}{\partial \bar{z}} \, d\bar{z} \otimes \frac{\partial}{\partial z}. \]

On the other hand, we have

\[ \bar{\partial} \xi = \frac{\partial f}{\partial \bar{z}} \, d\bar{z} \otimes \frac{\partial}{\partial z} \]

by definition of \( \bar{\partial} \) and hence the proof. \qed

Therefore the (Zariski) tangent space of the orbit \( \text{Orb}(j) \) of the action of \( \text{Diff}(\Sigma) \) is given by

\[ T_j \text{Orb}(j) = \{ \bar{\partial} \xi \mid \xi \in \Omega^0(T^{(1,0)}\Sigma) \}. \]  

(9.4.22)

In other words, in terms of the two-term complex

\[ 0 \to \Omega^0(T^{(1,0)}\Sigma) \to \Omega^0(T^{(0,1)}\Sigma) \to \Omega^0(T^{(1,0)}\Sigma) \to 0 \]

we have proved \( T_j \text{Orb}(j) = \text{im} \bar{\partial} \).

By combining the results of the above discussion, we obtain the following description of the tangent space of the moduli space, or infinitesimal deformation space, of complex structures.
Proposition 9.4.6 We have
\[ T_{[j]}(\mathcal{J}_\Sigma/\text{Diff}(\Sigma)) \cong H^1(T(1,0)\Sigma), \]
where the latter is the Dolbeault cohomology or the \( \bar{\partial}_j \) cohomology.

Remark 9.4.7 One can divide the action of \( \text{Diff}(\Sigma) \) into two stages. The first stage is the action of \( \text{Diff}_0(\Sigma) \), the identity component of \( \text{Diff}(\Sigma) \) and that of the quotient \( \text{Diff}(\Sigma)/\text{Diff}_0(\Sigma) = \text{Map}(\Sigma) \), the mapping class group of \( \Sigma \). Then the mapping class group \( \text{Map}(\Sigma) \) acts on the quotient \( T_g := \mathcal{J}_\Sigma/\text{Diff}_0(\Sigma) \), which induces the fibration
\[ T_g = \mathcal{J}_\Sigma/\text{Diff}_0(\Sigma) \leftarrow \text{Map}(\Sigma) \]
\[ \Downarrow \]
\[ \mathcal{M}_g(= \mathcal{J}_\Sigma/\text{Diff}(\Sigma)) \]
where \( T_g \) is called the Teichmüller space of genus-\( g \) Riemann surfaces. The space \( T_g \) is known to be contractible, while the topology of \( \mathcal{M}_g \) is more complicated because the action of the mapping class group on \( T_g \) is very non-trivial.

9.4.2 Deformations of marked Riemann surfaces
Now we consider a marked Riemann surface \((\Sigma, j, z)\) with
\[ z = \{z_1, z_2, \ldots, z_m\} \subset \Sigma. \]
\( \text{Diff}(\Sigma) \) acts on \((\Sigma, j, z)\) by the map
\[ (j, z) \rightarrow (\phi_* j, \phi(z)), \]
where \( \phi(z) = \{\phi(z_1), \ldots, \phi(z_m)\} \).

Definition 9.4.8 (Configuration space) For each \( m \geq 0 \), we denote
\[ \widehat{\text{Conf}}_m(\Sigma) = \Sigma^m \setminus \Delta^{\text{big}} \]
where the big diagonal \( \Delta^{\text{big}} \) is defined by
\[ \Delta^{\text{big}} = \{(z_1, \ldots, z_m) \in \Sigma^m \mid \exists (i, j), i < j, z_i = z_j\}. \]
The set \( \widehat{\text{Conf}}_m(\Sigma) \) is called the configuration space of \( m \) points on \( \Sigma \).
With this definition, the moduli space of stable curves with \( m \) marked points on \( \Sigma \) is given by

\[
M_{g,m} = J_\Sigma \times \widetilde{\text{Conf}}_m(\Sigma)/\text{Diff}(\Sigma).
\]

In this differential geometric point of view, we have the following.

**Definition 9.4.9** An automorphism of \(((\Sigma, j), z)\) is a diffeomorphism \( \phi : \Sigma \to \Sigma \) such that \( \phi_* j = j, \phi(z_j) = z_j, j = 1, \ldots, m \), i.e., a biholomorphism fixing the marked points. We define

\[
\text{Aut}((\Sigma, j), z) = \{ \phi \in \text{Diff}(\Sigma) \mid \phi_* j = j, \phi(z_j) = z_j, j = 1, \ldots, m \}
\]

\[
= \{ \phi \in \text{Aut}(\Sigma, j) \mid \phi(z_j) = z_j, j = 1, \ldots, m \}.
\]

A curve \(((\Sigma, j), z)\) is called **stable** if this group is finite. When there is no danger of confusion, we also just denote \( \text{Aut}(\Sigma, z) \) for \( \text{Aut}((\Sigma, j), z) \).

To describe the tangent space \( T_{[\Sigma, z]} M_{g,m} \), we first consider the tangent space

\[
T_{(j,z)} \left( J_\Sigma \times \widetilde{\text{Conf}}_m(\Sigma) \right)
\]

and the infinitesimal action of \( \text{Diff}(\Sigma) \) thereon.

Suppose that \((\eta, v) \in T_{(j,z)}(J_\Sigma \times \widetilde{\text{Conf}}_m(\Sigma))\). A diffeomorphism \( \phi \in \text{Diff}(\Sigma) \) induces the map

\[
(\eta, v) \mapsto (\phi \cdot \eta, d\phi(v)),
\]

where \( v = (v_1, \ldots, v_m) \in T_{z_1} \Sigma \times \cdots \times T_{z_m} \Sigma \).

Let us now give a concrete description of \( T_{[\Sigma, z]} M_{g,m} \). We recall from (9.4.22)

\[
T_{(j,z)} \text{Orb}(j, z) \cong \{ (\partial \xi, [\xi(z_j)]) \mid \xi \in \Omega^0(T^{(1,0)} \Sigma) \}.
\]

We denote by \( W^{1,p}(T^{(1,0)} \Sigma) \) the \( W^{1,p} \)-completion of \( \Omega^0(T^{(1,0)} \Sigma) \) and by \( L^p(\Lambda^{0,1}T^{(1,0)} \Sigma) \) the \( L^p \)-completion of \( \Omega^{0,1}(T^{(1,0)} \Sigma) \). We showed before that \( \partial : \Omega^0(T^{(1,0)} \Sigma) \to \Omega^{0,1}(T^{(1,0)} \Sigma) \) is an elliptic differential operator of order one whose symbol is the same as the standard Cauchy–Riemann operator. Thus \( \partial \)

extends to a continuous operator \( \partial : W^{1,p}(T^{(1,0)} \Sigma) \to L^p(\Lambda^{0,1}(T^{(1,0)} \Sigma)) \).

**Proposition 9.4.10** \( \partial : W^{1,p}(T^{(1,0)} \Sigma) \to L^p(\Lambda^{0,1}T^{(1,0)} \Sigma) \) is a Fredholm operator, and so is the linear map

\[
\partial_0 : W^{1,p}(T^{(1,0)} \Sigma) \to L^p(\Lambda^{0,1}T^{(1,0)} \Sigma) \oplus \bigoplus_{j=1}^m T_{z_j} \Sigma
\]

defined by \( \partial_0(\xi) = (\partial \xi, [\xi(z_j)]) \).
Exercise 9.4.11  Prove that the stability condition is equivalent to \( \ker \partial_0 = \{0\} \). (Hint. The proof of one direction is considerably harder to prove than the other.)

On the other hand, since \( \mathcal{M}_{g,m} = \mathcal{J}_\Sigma \times \widetilde{\text{Conf}}_m(\Sigma) / \text{Diff}(\Sigma) \), we have the isomorphism

\[
T_{[j,z]} \mathcal{M}_{g,m} \cong \text{coker } \partial_0. \tag{9.4.23}
\]

Remark 9.4.12  We have the following \( \text{Aut}(\Sigma, z) \)-equivariant fibration:

\[
(\xi, \{ v_j \}) \in T_{(j,z)}(\mathcal{J}_\Sigma \times \widetilde{\text{Conf}}_m(\Sigma)) \leftarrow \oplus_{j=1}^m T_{z_j} \Sigma
\]

\[
[x] \in H^1(T\Sigma) \cong T_{[j]} \mathcal{M}_g
\]

Here the actions of \( \text{Aut}(\Sigma, z) \) on \( \oplus_{j=1}^m T_{z_j} \Sigma \) and \( H^1(T\Sigma) \) are those canonically induced by the action on \( (\Sigma, z) \).

The above description of \( T_{[j,z]} \mathcal{M}_{g,m} \) is not very concrete in that an element thereof is a \( \text{Diff}(\Sigma) \)-orbit of the pair \( (\xi, \{ v_j \}) \) but not an element of a function space. It is often important, especially in relation to the gluing problem of pseudoholomorphic maps, to represent the tangent space as a subspace of \( T_{(j,z)}\mathcal{J}_\Sigma \times \widetilde{\text{Conf}}_m(\Sigma) \) that is transverse to the orbit \( T_{(j,z)} \text{Orb}(j, z) \) and invariant under the action of \( \text{Aut}(\Sigma, z) \).

For this purpose, we choose a Hermitian metric \( h \) on \( \Sigma \) that is invariant under the actions of \( \text{Aut}(\Sigma, z) \), which is certainly possible since \( \#(\text{Aut}(\Sigma, z)) \) is finite. Here, as the literature in complex geometry commonly does, we will identify \( (\mathcal{T}(\Sigma, j)) / \cong (\mathcal{J}(\Sigma, j, \sqrt{-1})) \)

as a holomorphic line bundle by the map

\[
X \mapsto X - \frac{\sqrt{-1}}{2} jX
\]

and denote by \( \xi \) the associated complex vector.

We then equip \( \Omega^{(0,1)}(T\Sigma) \oplus \bigoplus_{j=1}^m T_{z_j} \Sigma \) with the norm

\[
\| \eta, \{ v_j \} \| = \| \eta \|_p + \sum_j |v_j|
\]

induced by the Hermitian metric \( h \). We recall that we have the isomorphism

\[
\left( L^p(\Lambda^{(0,1)} T\Sigma) \right)^* \cong L^q(\Lambda^{(1,0)} T\Sigma); \quad \frac{1}{p} + \frac{1}{q} = 1
\]
via the pairing
\[
(\alpha_1 \otimes \xi_1, \alpha_2 \otimes \xi_2) \mapsto \frac{\sqrt{-1}}{2} \int_{\Sigma} \alpha_1 \wedge \overline{\alpha}_2 h(\xi_1, \xi_2)
\]
for the indecomposable elements
\[
\eta_i = \alpha_i \otimes \xi_i \in \Lambda^{(0,1)}(T^{(1,0)} \Sigma) \cong \Lambda^{(0,1)}(T \Sigma)
\]
for \(i = 1, 2\), where \(\alpha_i \otimes \xi_i \in L^p(\Lambda^{(0,1)}(T \Sigma))\). Denote by \(\bar{\partial}_0^\dagger\) the adjoint map of \(\bar{\partial}_0\)
\[
\bar{\partial}_0^\dagger : L^q(\Lambda^{(0,1)} T \Sigma) \oplus \bigoplus_{j=1}^{m} T_{z_j} \Sigma \to \left( W^{1,p}(T \Sigma) \right)^* \cong W^{-1,q}(T \Sigma). \tag{9.4.24}
\]
Then we may identify \(T_{[j,z]} M_{g,m} \cong \text{coker } \bar{\partial}_0\) with \(\ker \bar{\partial}_0^\dagger\). We would like to point out that an element of \(W^{-1,q}(T \Sigma)\) is not necessarily a function but is a distribution. Since \(1 < q < 2\) \((p > 2)\), \(W^{-1,q}(T \Sigma)\) allows a function with a logarithmic singularity of the type \(\ln |z|\) because \((\ln |z|)' = 1/|z|\) lies in \(L^q\) in two dimensions.

The following lemma follows from the definition of the adjoint map \(\bar{\partial}_0^\dagger\).

**Lemma 9.4.13** Let \((\eta, \{u_j\}) \in L^q(\Lambda^{(0,1)} T \Sigma) \oplus \bigoplus_{j=1}^{m} T_{z_j} \Sigma\). Then \((\eta, \{u_j\}) \in \ker \bar{\partial}_0^\dagger\) if and only if
\[
\int_{\Sigma} \langle \bar{\partial} \xi, \eta \rangle + \sum_{j=1}^{m} \langle \xi(z_j), u_j \rangle = 0
\]
for all \(\xi \in \Omega^0(T \Sigma)\), where \(<\cdot, \cdot>\) denotes the natural pairings.

Now we analyze the equation in the lemma further. Considering \(\xi\) vanishing near the \(z_j\), we have
\[
\int_{\Sigma \setminus \{z_1, \ldots, z_m\}} \langle \bar{\partial} \xi, \eta \rangle = 0
\]
on \(\Sigma \setminus \{z_1, \ldots, z_m\}\). Therefore, \(\eta\) satisfies \(\bar{\partial}_0^\dagger \eta = 0\), where
\[
\bar{\partial}_0^\dagger : L^q(\Lambda^{(0,1)} T \Sigma) \to \left( W^{1,p}(T \Sigma) \right)^*
\]
is the formal adjoint of \(\bar{\partial}_0^\dagger\) defined similarly to the definition of \(\bar{\partial}_0^\dagger\). We will show that \(\bar{\partial}_0^\dagger\) is again the first-order elliptic differential operator whose symbol is the same as \(\bar{\partial}\). By the general elliptic theory, any such solution is automatically smooth on \(\Sigma \setminus \{z_1, \ldots, z_n\}\). But we would like to emphasize that \(\eta\) may not be smooth near marked points \(z_i\).
Now, we derive the local formula for $\bar{\partial} \eta$. To analyze the behavior of $\eta$ near the marked points, it is easier to work with coordinate calculations. Let $z$ be a complex coordinate centered at a fixed marked point $z_j$. Then we can express the metric $h$ as

$$h = \frac{1}{2} h_{\bar{z} z} \, dz \, d\bar{z}$$

for some complex-valued function $h_{\bar{z} z}$. We write the local sections $\eta$ and $\xi$, and the tangent vectors $v_j$ as

$$\xi = a \frac{\partial}{\partial z}, \quad \eta = b \, d\bar{z} \otimes \frac{\partial}{\partial \bar{z}}, \quad v_j = c_j \frac{\partial}{\partial z} \bigg|_{z_j},$$

respectively, where $a, b$ are $\mathbb{C}$-valued functions defined on the coordinate neighborhood of $z$, and $c_j$ are constants. Furthermore, $\bar{\partial} \eta$ has the form $\bar{\partial} \eta = c \, \partial / \partial z$ for some $\mathbb{C}$-valued function $c$, and then we will obtain

$$\langle \xi, \bar{\partial} \eta \rangle = a \, c \, h_{\bar{z} z}.$$

(9.4.25)

We compute

$$\bar{\partial} \xi = \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial}{\partial z} \right) d\bar{z} = \frac{\partial a}{\partial \bar{z}} \frac{\partial}{\partial \bar{z}} \otimes d\bar{z}$$

and so

$$\langle \bar{\partial} \xi, \eta \rangle = \left( \frac{\partial a}{\partial \bar{z}} \frac{\partial}{\partial z}, b \frac{\partial}{\partial \bar{z}} \right) i \frac{1}{2} \, d\bar{z} \wedge dz$$

$$= \frac{\partial}{\partial \bar{z}} \left( a \bar{b} h_{\bar{z} z} \right) d\bar{z} \wedge \left( i \frac{1}{2} \, dz \right) - a \frac{\partial}{\partial \bar{z}} \left( \bar{b} h_{\bar{z} z} \right) d\bar{z} \wedge \left( i \frac{1}{2} \, dz \right)$$

$$= d \left( a \bar{b} h_{\bar{z} z} \frac{i}{2} \, dz \right) - a \frac{\partial}{\partial \bar{z}} \left( \bar{b} h_{\bar{z} z} \right) dA. \quad (9.4.26)$$

On comparing (9.4.25) with (9.4.26), we have derived

$$\bar{\partial} \eta = - \frac{1}{h_{\bar{z} z}} \frac{\partial}{\partial z} \left( b h_{\bar{z} z} \right) \frac{\partial}{\partial \bar{z}}.$$

Therefore, if $(\eta, \{v_j\}) \in \ker \bar{\partial}_0$, $\xi$ satisfies

$$\bar{\partial} \eta = 0 \quad (9.4.27)$$

on $\Sigma \setminus \{z_1, \ldots, z_m\}$. Now, by substituting this back into (9.4.26), we derive

$$\langle \bar{\partial} \xi, \eta \rangle = d \left( a \bar{b} h_{\bar{z} z} \frac{i}{2} \, dz \right).$$
From this equation and the integrability of \( \langle \partial \xi, \eta \rangle \), we derive
\[
\int_{\Sigma} \langle \partial \xi, \eta \rangle = \lim_{\epsilon \to 0} \int_{\Sigma \setminus \bigcup_{j=1}^{m} B_{\epsilon}(z_j)} \langle \partial \xi, \eta \rangle = -\lim_{\epsilon \to 0} \sum_{j=1}^{m} \int_{\partial B_{\epsilon}(z_j)} abh_{\xi} \frac{i}{2} \, dz = -\sum_{j=1}^{m} a(z_j) \left( \lim_{\epsilon \to 0} \int_{\partial B_{\epsilon}(z_j)} \overline{b} h_{\xi} \frac{i}{2} \, dz \right).
\]

Since (9.4.27) implies that the function \( \overline{b} h_{\xi} \) and hence \( z \overline{b} h_{\xi} \) is holomorphic on a punctured neighborhood of \( z_j \), the integral is in fact independent of the radius \( \epsilon \). This motivates us to introduce the following definition.

**Definition 9.4.14** Consider the stable curve \((\Sigma, z)\) with \( z = \{z_1, \ldots, z_m\} \). Let \( \eta \in \ker \overline{\partial} \subset L^q(\Lambda^{0,1}\Sigma \otimes T\Sigma) \) and consider \( z_j \in \Sigma \). Write \( \eta = b d\bar{z} \otimes \partial/\partial z \) in a complex coordinate \( z \) at \( z_j \). Then we define
\[
\text{Res}_{z_j}(\eta) = \left( \frac{1}{2i} \int_{\partial B_{\epsilon}(z_j)} b \overline{h_{\xi}} \, dz \right) \frac{\partial}{\partial z} \in T_{z_j}\Sigma
\]
and call it the residue of \( \eta \) at \( z_j \).

With this definition, we have proved the following proposition.

**Proposition 9.4.15** Let \( \eta \in \ker \overline{\partial} \). The vector
\[
\left( \frac{1}{2i} \int_{\partial B_{\epsilon}(z_j)} \overline{b h_{\xi}} \, dz \right) \frac{\partial}{\partial z} \in T_{z_j}\Sigma
\]
is well defined and independent of the choice of the complex coordinate \( z \) at \( z_j \). Furthermore, the map \( \text{Res}_{z_j}|_{\ker \overline{\partial}} \mapsto T_{z_j}\Sigma \) is a homomorphism.

**Proof** We have already shown that \( b \overline{h_{\xi}} \) is holomorphic on \( B_{\epsilon}(z_j) \setminus \{z_j\} \). We now claim that the function \( z b(z) \overline{h_{\xi}(z)} \) is in \( L^2 \) on \( B_{\epsilon}(z_j) \). Recall that \( \eta \in \ker \overline{\partial} \in L^q(\Lambda^{0,1}\Sigma \otimes T\Sigma) \) with \( 1 < q < 2 \). On the other hand, \( |z| \) lies in \( L^p \) for any \( p > 0 \) and in particular for \( p \) with \( 1/p + 1/q = 1 \). Therefore \( z b(z) \overline{h_{\xi}(z)} \) lies in \( L^2 \) as claimed. This implies that the function is smooth across \( z_j \) and hence holomorphic everywhere on \( B_{\epsilon}(z_j) \) by removal of singularity. Now the Cauchy integral formula proves that the integral
\[
\int b(z) \overline{h_{\xi}(z)} \, dz = \oint \frac{z b(z) \overline{h_{\xi}}}{z} \, dz
\]
does not depend on the choice of closed loop in $B_\epsilon(z_j)$ around $z_j$ whose winding number is 1, which finishes the proof. □

Therefore we obtain

$$0 = \int_\Sigma \langle \partial \xi, \eta \rangle + \sum_{j=1}^m \langle \xi(z_j), u_j \rangle = \sum_{j=1}^m a(z_j)(c_j + b_j)h_{z_j}(z_j),$$

where

$$c_j = \frac{1}{2i h_{z_j}(z_j)} \oint b h_{z_j} dz$$

(i.e., $\text{Res}_{z_j}(\eta) = c_j \partial/\partial z$).

This equation is true for all $\eta$ and thus for all $a$. Hence

$$\text{Res}_{z_j}(\eta) + u_j = 0.$$ 

We now summarize the above discussion in the following theorem.

**Theorem 9.4.16** Let $(j, z) \in \mathcal{J}_\Sigma \times \text{Conf}_m$. Then we have the isomorphism

$$T_{[j,z]}M_{g,m} = T_{[j,z]}(\mathcal{J}_\Sigma \times (\text{Conf}_m/\text{Diff}_0(\Sigma)))$$

$$\cong \left\{(\eta, \{u_j\}) : \bar{\partial}^* \eta = 0, \text{Res}_{z_j}(\eta) = -u_j \right\},$$

where $(\eta, \{u_j\}) \in L^q(\Lambda^{0,1}\Sigma \otimes T\Sigma) \oplus \bigoplus_{j=1}^m T_{z_j}\Sigma$.

Now consideration of the elliptic complex (9.4.24) via the Riemann–Roch formula gives rise to the following.

**Theorem 9.4.17** We have $\dim \mathcal{M}_{g,m} = 3g - 3 + m$.

**Remark 9.4.18** Here we emphasize that $(\eta, \{u_j\})$ are coupled to each other by the residue equation $\text{Res}_{z_j}(\eta) = -u_j$.

### 9.4.3 Deformations of nodal curves

In this subsection, $(\Sigma, z)$ will denote a connected stable curve, i.e., a union of compact Riemann surfaces without a boundary with nodal singularities, and $[\Sigma, z]$ the corresponding element in $\overline{M}_{g,m}$. We will describe the deformation of such curves in $\overline{M}_{g,m}$.

The infinitesimal deformation space of $\overline{M}_{g,m}$ at a nodal curve will be split into the part preserving the intersection pattern and tangential to the given stratum $M_{g,m}$ and the part transverse to the stratum.
For the part leaving $D$ unchanged, the infinitesimal deformation is given by the fiber sum of

$$
\bigoplus_{v \in V_T} T_{[\Sigma,z_v]}\mathcal{M}_{g,v,m_v}/\text{Aut}(\Sigma,z)
$$

over the matching condition

$$
\xi_v(x_v) = \xi_w(x_w)
$$

at all double points $x_e = (x_v, x_w)$. Here we recall that

$$
m_v = \text{sing}(v) + \text{mark}(v), \quad g_v = \text{genus}(\Sigma_v).
$$

We can encode this into the following exact sequence:

$$
0 \to T_{[\Sigma,z]}\mathcal{M}_{g,m}^D \to \bigoplus_{v \in V_T} T_{[\Sigma,z_v]}\mathcal{M}_{g,v,m_v} \to \bigoplus_{e \in E_T} T_{x_e}\Sigma_{x_e} \to 0, \quad (9.4.28)
$$

where the second map is the map

$$
\{\xi_v\}_{v \in V_T} \mapsto \{\xi_v(x_v) - \xi_w(x_w)\}_{e \in E_T}.
$$

In particular, we obtain the following dimension formula for $\mathcal{M}_{g,m}^D$.

**Proposition 9.4.19** Let $D \in \text{Comb}(g, m)$. Then we have

$$
\dim \mathcal{M}_{g,m}^D = 3g + m - 3 - \#(\text{of double points}). \quad (9.4.29)
$$

**Proof** It follows from (9.4.28) that

$$
\dim \mathcal{M}_{g,m}^D = \sum_{v \in V_T} \dim T_{[\Sigma,z_v]}\mathcal{M}_{g,v,m_v} - \sum_v \text{sing}(v)
$$

$$
= \sum_{v \in V_T} (3g_v - 3 + m_v) - \sum_v \text{sing}(v)
$$

$$
= 3 \sum_v g_v - 3\#(V_T) + \sum_v \text{mark}(v) + 2 \sum_v \text{sing}(v) - \sum_v \text{sing}(v)
$$

$$
= 3(g - b_1(T)) - 3\#(V_T) + m + \sum_v \text{sing}(v)
$$

$$
= 3g + m - 3(1 - \#(V_T) + \#(E_T)) - 3\#(V_T) + 2\#(E_T)
$$

$$
= 3g + m - 3 - \#(E_T).
$$

Here the term $2 \sum_v \text{sing}(v)$ comes from the fact that each double point is counted twice in the sum $\sum_v m_v$. This finishes the proof. \qed

Next, we consider the deformations transverse to the stratum $\mathcal{M}_{g,m}^D$, i.e., those for which some of the nodal points are resolved. We will describe this deformation in two steps: the first is smoothing of singular points of the nodal
surface as a space and the second is gluing of complex structures. We obtain both by gluing either manifolds or conformal structures. In particular, we will realize the glued conformal structure by gluing out of a suitable family of Kähler metrics following the prescription given by Fukaya and Ono (FOn99).

We start with the following general smooth description of the deformation of nodal curves following (HWZ02) and (FOn99), because it well suits the purpose of gluing of pseudoholomorphic curves. We refer readers to any algebraic geometry literature, e.g., (DM69), for the precise algebraic description of the general deformation theory of algebraic curves.

**Definition 9.4.20** (Definition 4.1, (HWZ02)) A deformation of a compact Riemann surface $(A, j)$ of annulus type is a continuous surjection map $f : A \to S$ onto the nodal surface, so that $f^{-1}(o)$ is a smooth embedded circle, and

$$f : A \setminus f^{-1}(o) \to S \setminus \{o\}$$

is an orientation-preserving diffeomorphism. On $S \setminus \{o\}$ we have the push-forward complex structure $f_*j$. We call $S$ an annular nodal curve and $o \in S$ a nodal point.

For each given nodal surface $S$, we recall the construction of a family of deformations in the following way (see (FOn99)) parameterized by $\alpha \in \mathbb{C}$ with $|\alpha|$ sufficiently small. We call this explicit deformation Fukaya–Ono deformation to emphasize the importance of this kind of particular choice of parameters in the gluing process. Such a choice is necessary in order to guarantee the a-priori estimates needed in the gluing process.

**Example 9.4.21** (Fukaya–Ono deformation, (FOn99)) We choose the unique biholomorphic map

$$\Phi_\alpha : T_\alpha S_- \setminus \{o_-\} \to T_\alpha S_+ \setminus \{o_+\},$$

such that $u \otimes \Phi_\alpha(u) = \alpha$. In terms of analytic coordinates at $o_- \in S_-$ and $o_+ \in S_+$, the coordinate expression of $\Phi_\alpha$ is given by the map $\Phi_\alpha(z) = \alpha/z$.

We denote $|\alpha| = R_\alpha^{-1}$ for $|\alpha|$ sufficiently small and hence $R_\alpha$ sufficiently large so that the composition

$$\exp_{S_-}^{-1} \circ \Phi_\alpha \circ \exp_{S_+} : D_{o_-}(R_\alpha^{-1/2}) \setminus D_{o_-}(R_\alpha^{-3/2}) \to D_{o_+}(R_\alpha^{-1/2}) \setminus D_{o_+}(R_\alpha^{-3/2})$$

is a diffeomorphism (see Figure 9.5). By composing this map with the biholomorphism

$$[-\ln R_\alpha^{-1/2}, \ln R_\alpha^{-1/2}] \times S^1 \to D_{o_+}(R_\alpha^{-1/2}) \setminus D_{o_+}(R_\alpha^{-3/2});$$

$$(\tau, t) \mapsto e^{2\pi((\tau-R)+it)} = e^{-2\pi R} z$$
with \( z = e^{2\pi(\tau + it)} \) the standard coordinate on \( \mathbb{C} \), this diffeomorphism becomes nothing but

\[
[-\ln R^{-1/2}_\alpha, \ln R^{1/2}_\alpha] \times S^1 \rightarrow [-\ln R^{-1/2}_\alpha, \ln R^{1/2}_\alpha] \times S^1;
\]

\((\tau, t) \mapsto (-\tau, -t) = (\tau', t').\)

See Figure 9.6.

We glue the metrics on

\[ D_{o_\alpha}(R^{-1/2}_\alpha) - D_{o_\alpha}(R^{-3/2}_\alpha) \]

without changing the metric outside \( D_{o_\alpha}(R^{-1/2}_\alpha) \) on \( \Sigma_0 \). Identify \( D_{o_\alpha}(R^{-1/2}_\alpha) \) with an open set in \( \mathbb{C} \ni z \) with the standard metric. Consider the biholomorphism \( \Phi_\alpha : z \rightarrow \alpha/z \), for which we have

\[
(\Phi_\alpha)^*|dz|^2 = \left| \frac{\alpha}{z^2} \right|^2 |dz|^2.
\]

Note that, on \(|z| = R^{-1}\), we have

\[
\Phi_\alpha(|z|z) = \sqrt{\alpha},
\]

\[
(\Phi_\alpha)^*|dz|^2 = |dz|^2.
\]

We choose a function

\[
\chi_{R_\alpha} : (0, \infty) \rightarrow (0, \infty)
\]
and fix it once and for all such that

1. \((\Phi_\alpha)^*(\chi_{R_\alpha}|dz|^2) = \chi_{R_\alpha}|dz|^2\)
2. \(\chi_{R_\alpha}(r) \equiv 1\) if \(r > |\alpha|^{3/8} = R_{\alpha}^{-3/4}\).

By virtue of the definition of \(\chi_{R_\alpha}\), we can replace the given metric \(g_{o_{\alpha}} = |dz|^2\) by \(\chi_{R_\alpha}|dz|^2\) inside the disc \(D^2(|\alpha|^{1/4})\), and denote the resulting metric by \(g'_{o_{\alpha}}\). We would like to emphasize that this modification process is canonical, depending only on the fixed complex charts at the singular points and on the choice of \(\chi_{R_\alpha}\). As a result, this modification process does not add more parameters to the description of deformation of stable curves. Hence we have constructed a family of stable curves parameterized by a neighborhood of the origin in \(T_o, S_+ \otimes T_o, S_-\). We denote the Riemann surface with the conformal structure constructed in this way by

\[(S_\alpha, j_\alpha)\].

We set \(S_0\) to be the given nodal Riemann surface \(S\). We can define a surjective continuous map \(f_\alpha : S_\alpha \to S\) by the projection from the graph of \(w = \alpha/z\) to the union of the \(z\)-axis and \(w\)-axis that is invariant under the diagonal reflection.

This finishes the construction of the one-parameter family of deformations of the given nodal Riemann surface.

For each nodal point of \(\Sigma\), we denote by \(e\) the corresponding edge in the dual graph \(T\). Then we denote the double point by \(x_e = (x_v, x_w)\), where \(e = [v, w]\) with \(e \in E_T, v, w \in V_T\) for the dual graph \(T\) of the stratum \(D\).

We fix a complex chart \(z : U_v \to \mathbb{C}\) centered at \(x_v \in U_v \subset \Sigma_v\) at each singular point of \(\Sigma_v\) for all \(v \in V_T\). We will also denote \(U_v = U_{v, e}\) when we need to highlight the edge containing the vertex \(v\).

We choose a constant \(\epsilon_0 = \epsilon_0(\Sigma, z) > 0\) and fix it once and for all so that

\[z(U_v) \supset D^2(\epsilon_0)\]

for all \(v \in V_T\) and \(e \in E_T\), and fix a Kähler metric \(h = \{h_v\}\) on \(\Sigma\) so that

1. \(h\) is invariant under the action of \(\text{Aut}(\Sigma, z)\),
2. each \(h_v\) coincides with the pull-back of the standard metric of \(D^2(\epsilon_0) \subset \mathbb{C}\) on \(z^{-1}(D^2(\epsilon_0))\) and
3. \(h_v\) represents the conformal structure \(j_v\) for each \(v\).

The existence of such metrics is a consequence of the existence theorem of isothermal coordinates (Kor16, Li16) (see, e.g., (Cher55) for a later account).

With these preparations, we are now ready to carry out the gluing process. First, gluing of the complex structures will be done by the map
in the given complex coordinate charts. Using this diffeomorphism, we can apply the construction of the Fukaya–Ono deformation of the annular nodal curve $\Sigma_v \cup \Sigma_w$. By performing this construction at each singular point, we obtain a two-dimensional ‘manifold’ for each element $(a_i) \in \bigoplus_x \lambda \Sigma_v \otimes T_x \Sigma_w$ in a neighborhood of 0. The resulting manifold will be singular when some $a_i = 0$. (We leave the singular point untouched in this case.)

However, in order to construct a coordinate (orbifold) chart at a nodal curve, we need to make this gluing process more precise by describing the gluing parameters precisely. Furthermore, we also need to establish the compatibility of the charts which have been constructed pointwise. For this purpose, we need to carry out our construction inductively, starting from the minimal strata.

Note that $\Phi_a((z|z = \sqrt{|a|})) = |z|z = \sqrt{|a|}$. So $\Phi_a$ is the standard reflection along the circle of radius $\sqrt{|a|}$.

**Step I. Minimal stratum**

Consider a minimal stratum $D$ with respect to the partial order given in Definition 9.3.16. For any element $(\Sigma, z)$ in any such stratum $D$ with $\Sigma = \bigcup_v \Sigma_v$, we have $g_v = 0$ and $m_v = 3$ for each $v$. See Figure 9.7.

Therefore $M_{g,m}^0$ contains a single element for any intersection pattern $D$ of a connected component of the minimal strata. Recall that we have chosen a Kähler metric $h_v$ on each $\Sigma_v$, so that $h_v$ is invariant under $\text{Aut}(\Sigma, z)$ and flat in a neighborhood of each singular point $x_v$, so the exponential map is an isometry.

For each given edge $e \in E_T$ with its vertices $v$ and $w$, and an element $a \in T_{x_v} \Sigma_v \otimes T_{x_w} \Sigma_w \setminus \{0\}$, we choose the unique biholomorphic map

$$\Phi_a : T_{x_v} \Sigma_v \setminus \{0\} \to T_{x_w} \Sigma_w \setminus \{0\}$$
such that $u \otimes \Phi_a(u) = a$. (In the given coordinates at $x_v \in \Sigma_v$ and $x_w \in \Sigma_w$, its coordinate expression is precisely the map $\Phi_a(z) = a/z$.)

We would like to emphasize that Fukaya–Ono deformation is canonical, depending only on the fixed complex charts at the nodal points and on the choice of cut-off functions $\chi_R$. As a result, this modification process does not add more parameters to the description of the deformation of stable curves. Hence we have constructed a family of stable curves parameterized by a neighborhood of the origin in $\bigoplus_e (T_{x_v} \Sigma_v \otimes T_{x_w} \Sigma_w)$. We denote the surfaces obtained by Fukaya–Ono deformations by

$$\text{Res}_{(\Sigma, z)}([\alpha_e])_{e \in E_T},$$

where $D = (T, (g_v), o)$ is the dual graph of $(\Sigma, z)$. An automorphism, say $\gamma$, of $(\Sigma, z)$ acts by isometry on $\bigoplus_{(v, w) = e \in E_T} T_{x_v} \Sigma_v \otimes T_{x_w} \Sigma_w$. Obviously, the pre-stable curves corresponding to $a_x$ and $\gamma(a_x)$ are isometric. Hence, we have a family of elements of $\overline{M}_{g,m}$ parameterized by the neighborhood of 0 in

$$\bigoplus_{e = (v, w)} (T_{x_v} \Sigma_v \otimes T_{x_w} \Sigma_w)_{\text{Aut}(\Sigma, z)}$$

We summarize the above discussion in the following.

**Proposition 9.4.22** The map, constructed by Fukaya–Ono deformations,

$$\bigoplus_{(v, w) = e \in E_T} (T_{x_v} \Sigma_v \otimes T_{x_w} \Sigma_w)_{\text{Aut}(\Sigma, z)} \to \overline{M}_{g,m}$$

is one-to-one and onto its image on a neighborhood of

$$[0] \in \bigoplus_{e} (T_{x_v} \Sigma_v \otimes T_{x_w} \Sigma_w)_{\text{Aut}(\Sigma, z)},$$

and hence provides an orbifold chart of $\overline{M}_{g,m}$ at a point $[\Sigma, z]$ contained in a minimal stratum.

The proof of (local) surjectivity is quite non-trivial and goes beyond the scope of this book. (See (DM69) for an algebraic proof and (RS06) for a smooth proof.) We leave it as a challenging exercise.

**Exercise 9.4.23** Complete the details of the proof of this proposition, especially that of local surjectivity.

A precise discussion of orbifolds and a complete description of the orbifold structure of normal cones of $M^D_{g,m}$ in $\overline{M}_{g,m}$ as a stratified manifold (or rather a stratified orbifold) requires much study, which goes beyond the scope of this
9.4 Deformations of stable curves

Deformations of stable curves

\[ \mathcal{D}^g,m \]

Figure 9.8 Extension of neighborhoods.

We refer readers to the original article (DM69) or the book (MFK94) for a complete description of \( \overline{M}_{g,m} \) as a Deligne–Mumford stack.

**Step II. General strata**

Suppose we have constructed a family of metrics of each stable curve in a neighborhood \( \mathcal{N}_g^m \) of \( \mathcal{M}_g^m \), with \( D' < D \), which coincides on the overlaps of the previously chosen local neighborhoods of \( \mathcal{N}_g^m \). With respect to the given topology on the stratum \( \mathcal{M}_g^m \), choose a compact subset \( \mathcal{K}_g^m \subset \mathcal{M}_g^m \), so that

\[ \mathcal{M}_g^m \subset \mathcal{K}_g^m \cup \bigcup_{D' \leq D} \mathcal{N}_g^m. \]

See Figure 9.8.

We have the universal fibration

\[
\begin{array}{ccc}
\mathcal{K}_g^m & \xleftarrow{\scriptstyle \oplus} & \mathcal{N}_g^m \\
\mathcal{K}_g^m & \xrightarrow{\scriptstyle \oplus} & \mathcal{M}_g^m \\
\mathcal{K}_g^m & \xrightarrow{\scriptstyle \oplus} & \mathcal{M}_g^m \\
\end{array}
\]

which is the restriction of the forgetful maps \( \overline{M}_{g,m+1} \to \overline{M}_{g,m} \) and is restricted to a smooth fibration on \( \mathcal{K}_g^m \). In fact, the fiber can be parameterized fiberwise by \( \bigoplus_e (T_{x_v} \Sigma_v \otimes T_{x_w} \Sigma_w) / \text{Aut}(\Sigma, z) \) as in the same construction in Step I.

**Exercise 9.4.24** Fill in the details of this statement.
Now we perform the same constructions as in Step I fiberwise along the normal direction to the stratum \( \mathcal{M}_{g,m}^D \) in \( \overline{\mathcal{M}}_{g,m} \) over the compact subset \( \mathcal{K}_{g,m}^D \subset \mathcal{M}_{g,m}^D \), which extends the metrics already present in the overlap
\[
\mathcal{K}_{g,m}^D \cap \bigcup_{D' \leq D} \mathcal{N}_{g,m}^{D'}.
\]
We denote by \( U_D \) the set of stable curves represented by \( (\Sigma, z) \in \mathcal{M}_{g,m} \) and denote
\[
\Phi_D : U_D \times \bigoplus_{v \in E_T} \frac{T_x \Sigma_v \otimes T_x \Sigma_w}{\text{Aut}(\Sigma, z)} \to \overline{\mathcal{M}}_{g,m}.
\]
In this way, we have constructed an atlas of \( \overline{\mathcal{M}}_{g,m} \) consisting of orbifold charts, given by the images of \( \Phi_D \), where \( U^D \subset \mathcal{M}_{g,m}^D \) is an orbifold neighborhood. Therefore, we have produced an atlas and hence have finally provided a topology on \( \overline{\mathcal{M}}_{g,m} \).

**Exercise 9.4.25** Prove that these charts are compatible (in the sense of orbifolds).

The following proposition describes the structure of normal cones of \( \mathcal{M}_{g,m}^D \) in \( \overline{\mathcal{M}}_{g,m} \) as a stratified manifold (or rather a stratified orbifold). Again, a completely precise description of this normal cone is beyond the scope of this book, so we refer the reader to the book (MFK94). For the main purpose of this book, it will suffice to present the precise statement and then just outline what goes into its proof here.

**Proposition 9.4.26** \( \mathcal{M}_{g,m}^D \) is an orbifold and carries a fibration \( \mathcal{U}_{g,m}^D \to \mathcal{M}_{g,m}^D \) whose fiber at \( (\Sigma, z) \) is \( (\Sigma, z)/\text{Aut}(\Sigma, z) \).

**Outline of the proof** Let \( \Sigma = \bigcup_v \Sigma_v \in \mathcal{M}_{g,m}^D \). For each \( \Sigma_v \), denote by \( z_v \) the set of special points and by \( m_v \) its cardinality, which is nothing but \( \text{mark}(v) + \text{sing}(v) \).

Firstly, we look at a neighborhood of each point lying in a minimal stratum and construct a coordinate chart for it. Denote by \( \mathcal{V}(\mathcal{M}_{g,m}^{\text{min}}) \) the union of these charts which provides an open neighborhood of the minimal strata denoted by \( \mathcal{M}_{g,m}^{\text{min}} \). Since the minimal stratum is compact, we can cover it by a finite number of such product charts. Secondly, we consider a neighborhood of points from the next stratum, say \( D_1 \), away from the minimal stratum. We can construct a chart by taking a product chart of the given minimal stratum and a small normal
slice of the stratum at the given point. Here, in this process, it is important to observe that the complement

\[ \mathcal{M}^{D_1}_{g,m} \setminus \mathcal{V}(\overline{\mathcal{M}}_{g,m}^{\text{min}}) \]

is compact and hence we can again cover this complement by a finite number of such product charts. We will study the total deformation of \((\Sigma, z_v)\) and explain that it is modeled by \(\mathbb{C}^{3g-3+m}/\text{Aut}(\Sigma, z_v)\) in the next section, which enables us to choose an orbifold chart in the normal slice. This provides the structure of normal cones to the minimal strata. Once we have established this initial step, the picture in this second step resembles the family version of the picture for the minimal stratum considered in the first step. So we repeat the same process as the first over the family \(\mathcal{M}^{\text{min}}_{g,m}\).

We then proceed inductively over the given partial order of the set of dual graphs \(D\). On collecting all the coordinate charts, we have provided an (orbifold) atlas with \(\overline{\mathcal{M}}_{g,m}\).

We finally state the following basic theorem on the topology of \(\overline{\mathcal{M}}_{g,m}\) and provide an outline of its proof by the same token as Proposition 9.4.26. (We refer interested readers to the original article (DM69) for an algebraic proof and (RS06) for an account in the smooth context.)

**Theorem 9.4.27** (Deligne–Mumford) \(\overline{\mathcal{M}}_{g,m}\) with this topology is compact and Hausdorff.

**Outline of the proof** We first prove the compactness. Let \([\Sigma_i, z_i]\) be a sequence in \(\overline{\mathcal{M}}_{g,m}\). Since there are only finitely many topological types of the dual graphs by Theorem 9.3.14, we may assume that \([\Sigma_i, z_i]\) all lie in \(\mathcal{M}^D_{g,m}\) for the same \(D\) by choosing a subsequence if necessary.

If some subsequence of \([\Sigma_i, z_i]\) remains in some compact subset \(\mathcal{K}^D_{g,m} \subset \mathcal{M}^D_{g,m}\) for some \(D\), we are done. Otherwise Proposition 9.3.19 says that either two marked points should collide or the complex structure degenerates. The following lemma is needed for a complete proof. Since the proof of this lemma goes beyond the scope of this book, we leave its proof to (DM69) and (RS06).

**Lemma 9.4.28** Any degeneration sequence contains a subsequence that approaches some stratum \(\mathcal{M}^{D'}_{g,m}\) for some \(D' < D\).

Therefore Proposition 9.4.22 implies that we can write

\[(\Sigma_i, z_i) = \text{Res}_{(\Sigma'_i, z'_i)}(\{\alpha_{e,i}'\}_{e \in E_T}),\]

where \((\Sigma'_i, z'_i) \in \mathcal{M}^{D'}_{g,m}\) and \(|\alpha_{e,i}'| \to 0\) by Proposition 9.4.22. If \([\Sigma'_i, z'_i]\) has a convergent subsequence in \(\mathcal{M}^{D'}_{g,m}\), we are done. Otherwise we apply the above
process to the sequence \((\Sigma_i', z_i')\). Since there are only finitely many types of \(D\), after a finite number of steps, we can find a subsequence that converges in the topology of \(\overline{M}_{g,m}\). This finishes the proof of compactness.

Next we study the Hausdorff property of \(\overline{M}_{g,m}\). Noting that the set of stable curves \((\Sigma, z)\) (not of the equivalence classes \([\Sigma, z]\) at the moment) is a metric space, we denote the corresponding distance function thereon by \(\text{dist}\). We will prove the Hausdorff property by contradiction.

Suppose to the contrary that there exists a pair of stable curves \((\Sigma, z)\) \(\neq (\Sigma', z')\) such that \([\Sigma, z] \neq [\Sigma', z']\) but there exist a sequence \((\Sigma_i, z_i)\) and \(\phi_i, \phi'_i \in \text{Aut}(\Sigma_i, z_i)\) such that

\[
\text{dist}_{C^1}(\phi_i(\Sigma_i, z_i), (\Sigma, z)) \to 0,
\]

\[
\text{dist}_{C^1}(\phi'_i(\Sigma_i, z_i), (\Sigma', z')) \to 0.
\]

First we consider the case where \([\Sigma, z]\) and \([\Sigma', z']\) are in the same stratum \(\mathcal{M}^D_{g,m}\).

If both limits \((\Sigma, z)\), \((\Sigma', z')\) lie in the same stratum as that of \((\Sigma_i, z_i)\), then, after choosing a subsequence, we can represent \([\Sigma_i, z_i]\) by a single nodal surface \(\Sigma\) so that

\[
\phi_i((\Sigma_i, j_i), z_i) = ((\Sigma, \tilde{j}_i), x_i)
\]

and \((\tilde{j}_i, x_i) \to (j, z)\). Similarly, we have

\[
\phi'_i((\Sigma_i, j_i), z_i) = ((\Sigma', \tilde{j}'_i), x'_i)
\]

and \((\tilde{j}'_i, x'_i) \to (j', z')\). Therefore the map

\[
\psi_i := \phi'_i \circ \phi^{-1}_i : \Sigma \to \Sigma'
\]

satisfies

\[
(\psi_i, \tilde{j}_i \to \tilde{j}'_i, \quad \psi_i(x_i) \to x'_i)
\]

and \(\psi_i\) is a \((\tilde{j}_i, \tilde{j}'_i)\)-holomorphic map.

**Lemma 9.4.29** The sequence \(\psi_i\) is equi-continuous and hence carries a convergent subsequence in \(C^\infty\) topology.

**Proof** It suffices to prove the easy version of the derivative bound \(|d\psi_i|_{C^1} < C\) on each irreducible component of \(\Sigma\), since then higher regularity will follow from the main estimate, Theorem 8.3.5. Suppose to the contrary that \(|d\psi_i(z_i)| \to \infty\) as \(i \to \infty\) for some subsequence \(\psi_i\) and \(z_i \in \Sigma\). We have shown in Section 9.2 that if \(|d\psi_i(z_i)| \to \infty\) as \(i \to \infty\) then we can produce a non-constant \((j, j')\)-holomorphic sphere \(v : S^2 \to \Sigma'\) and hence \(\int v^* \omega' > 0\).
If genus(Σ') ≥ 1, π₂(Σ') = 0 and hence \( \int v^* \omega' > 0 \) gives rise to a contradiction.

It remains to consider the case genus(Σ') = 0. In this case we also have genus(Σ) = 0, the complex structure is unique and hence we can identify (Σ', j') \( \cong \mathbb{C}P^1 \cong (\Sigma, j) \). Therefore each \( \psi_i \) is a biholomorphism and thus lies in \( PSL(2, \mathbb{C}) \). On the other hand, the stability condition then implies \( \#(z_i) = \#(z'_i) \geq 3 \). Using the convergence of \((\tilde{j}', x'_i) \to (j', z')\) and \((\tilde{j}, x_i) \to (j, z)\) and applying an automorphism of domain \( \mathbb{C}P^1 \), we may assume \( z_{i,1} = 0, z_{i,2} = 1, z_{i,3} = \infty \) and

\[
\text{dist}(\psi_i(0), \psi_i(1)), \text{dist}(\psi_i(0), \psi_i(1)), \text{dist}(\psi_i(0), \psi_i(1)) \geq \delta > 0.
\]

This implies that the sequence \( \psi_i \in PSL(2, \mathbb{C}) \) is pre-compact and carries a subsequence that converges to an isomorphism between \((\mathbb{C}P^1, z)\) and \((\mathbb{C}P^1, z')\). This gives rise to a contradiction to the hypothesis \([\Sigma, z] \neq [\Sigma', z']\) and finishes the proof. \( \square \)

Now we consider the case where \((\Sigma, z), (\Sigma', z')\) are in a stratum \( D' \) different from the stratum \( D' \) of \([\Sigma_i, z_i] = M_{g,m}^{D'} \). In this case, we can represent

\[
\phi_i((\Sigma_i, j_i), z_i) = \text{Res}_{(\Sigma, z)}(\alpha_i), \\
\phi'_i((\Sigma_i, j_i), z_i) = \text{Res}_{(\Sigma', z')}(\alpha'_i)
\]

with \(|\alpha_i|, |\alpha'_i| \to 0\). After choosing a subsequence, we may assume that \( \Sigma_i = \Sigma'' \) for all \( i \). Then since \(|\alpha_i|, |\alpha'_i| \to 0\), \( \phi_i \circ \phi^{-1}_i : \Sigma'' \to \Sigma'' \) gives rise to a biholomorphism \( \psi : (\Sigma, z) \to (\Sigma', z') \) such that \( \psi(\Sigma, z) = (\Sigma', z') \) and \(|\alpha_i - \alpha'_i| \to 0\) as \( i \to \infty \). This again contradicts the statement \([\Sigma, z] \neq [\Sigma', z']\).

Finally, it remains to consider the case in which \([\Sigma, z] \) and \([\Sigma', z']\) lie in two different strata. This case is even easier to handle than the previous case and hence is left to the reader. \( \square \)

**Exercise 9.4.30** Complete the above proof by providing details of the proof for the case where \([\Sigma, z] \) and \([\Sigma', z']\) lie in two different strata.

### 9.5 Stable map and stable map topology

Let \( (M, \omega) \) be a compact symplectic manifold. Let \( J \) be an almost-complex structure compatible with \( \omega \). Let \( \Sigma \) be a smooth compact surface without a boundary. Let \( u : (\Sigma, j) \to (M, J) \) be a \( J \)-holomorphic map in the class \([u] = \beta \in H_2(M, \mathbb{Z})\) for a complex structure \( j \) on \( \Sigma \).
Now, we consider $\Sigma$ with marked points $z = \{z_1, z_2, \ldots, z_m \}$. As before, we just denote by $(\Sigma, z)$ the triple $((\Sigma, j), z)$ whenever there is no danger of confusion.

**Definition 9.5.1** Let $\Sigma$ and $\Sigma'$ be closed Riemann surfaces and $u : (\Sigma, z) \to M$, $u' : (\Sigma', z') \to M$ be $J$-holomorphic maps. We say that $u$ and $u'$ are equivalent if there exists an isomorphism $\phi : (\Sigma, z) \to (\Sigma', z')$ such that $u' = u \circ \phi^{-1}$ and $\phi(z_i) = \phi(z'_i)$. An automorphism $\phi$ of $((\Sigma, z), u)$ is a self-isomorphism of $((\Sigma, z), u)$. We call $((\Sigma, z), u)$ stable if $\# \text{Aut}((\Sigma, z), u) < \infty$.

Let $\widetilde{M}_{g,m}(M, J; \beta)$ be the set of smooth stable maps in class $\beta$ with $g(\Sigma) = g$, $m$ = the number of marked points.

Note that biholomorphisms $\phi : \Sigma \to \Sigma$ act on $\widetilde{M}_{g,m}(M, J; \beta)$ by

$$((\Sigma, z), u) \mapsto (\Sigma, \phi(z)), u \circ \phi^{-1},$$

where $\phi(z) = \{\phi(z_1), \ldots, \phi(z_m)\}$. The automorphism group $\text{Aut}((\Sigma, z), u)$ is nothing but the isotropy group of this action. We define

$$M_{g,m}(M, J; \beta) = \widetilde{M}_{g,m}(M, J; \beta)/\text{Aut}((\Sigma, z), u).$$

### 9.5.1 Definition of stable maps

Now we define the notion of stable maps of $J$-holomorphic maps defined on pre-stable curves.

Recall that a map $\phi : (\Sigma, z) \to (\Sigma', z')$ between two pre-stable curves is called an isomorphism if it is a homeomorphism and the restriction map $\phi_v : \Sigma_v \to \Sigma'$ can be lifted to a biholomorphism $\phi_{v, v'} : (\Sigma_v, z_v) \to (\Sigma'_v, z'_v)$ for some irreducible component $\Sigma'_v$ of $\Sigma'$. Here $z_v$ is the union of the marked points on $\Sigma_v$ and the singular points of $\Sigma_v$.

**Definition 9.5.2** Let $\Sigma$ be a nodal Riemann surface. A continuous map $u : \Sigma \to (M, J)$ is said to be $J$-holomorphic if it is continuous and the composition $u \circ \pi_v : \Sigma_v \to M$ is $J$-holomorphic for each $v$. Its homology class is given by

$$u_*([\Sigma]) = \sum_v (u \circ \pi_v)_*([\Sigma_v]) \in H_2(M, \mathbb{Z}).$$
Definition 9.5.3 Two marked $J$-holomorphic maps $((\Sigma, z), u)$, $((\Sigma', z'), u')$ are said to be isomorphic, if there exists an isomorphism $\phi : (\Sigma, z) \to (\Sigma', z')$ such that $u' = u \circ \phi^{-1}$. A self-isomorphism $\phi : (\Sigma, z) \to (\Sigma, z)$ is called an automorphism of $(\Sigma, z)$ if $u = u \circ \phi^{-1}$. We denote
\[
\text{Aut}((\Sigma, z), u) = \{ \phi \in \text{Aut}(\Sigma, z) | u \circ \phi = u \}.
\]
We call the pair $((\Sigma, z), u)$ a stable map if $\# \text{Aut}((\Sigma, z), u)$ is finite.

Remark 9.5.4
(1) Note that when $((\Sigma, z), u)$ and $((\Sigma', z'), u')$ are isomorphic a choice of isomorphism $\phi: (\Sigma, z) \to (\Sigma', z')$ between them induces the canonical isomorphism between the two groups $\text{Aut}((\Sigma, z), u)$ and $\text{Aut}((\Sigma', z'), u')$ by conjugation by $\phi$.

(2) We note that $\text{Aut}((\Sigma, z), u)$ is a subset of the product of the symmetry group $\text{Sym}(D)$ of the dual graph $D$ and the fiber product
\[
\prod_{v \in V^s_T} \text{Aut}((\Sigma_v, z_v), u_v) \times \prod_{v \in V^a_T} \text{Aut}((\Sigma_v, z_v), u_v)
\]
under the matching conditions $\phi_v(x_v) = \phi_w(x_w)$ for each element $e = [v, w] \in E_T$. Here $V^s_T$ and $V^a_T$ are the sets of vertices consisting of stable and unstable ones, respectively. Obviously we have $\# \text{Aut}(D) < \infty$.

Definition 9.5.5 We define the moduli space of stable maps to be the set of isomorphism classes of stable maps and denote it by
\[
\overline{M}_{g,m}(M, J; \beta).
\]
We examine a few special cases.

Example 9.5.6
(1) Consider the case $\beta = 0$. In this case, any $J$-holomorphic map in the class must be constant and hence its domain must be a stable curve. Therefore we have the one-to-one correspondence
\[
\overline{M}_{g,m}(M, J; 0) \simeq \overline{M}_{g,m} \times M.
\]

(2) Next we consider the case $\beta \neq 0$ and $g = 0$. In this case, the uniformization theorem implies that $\Sigma$ has the unique conformal structure (or complex structure). Therefore the space $\mathcal{M}_{0,m}(M, J; \beta)$ is isomorphic to
\[
\{(u, z) | u : \mathbb{C}P^1 \to M, \bar{\partial} u = 0, \quad z = (z_1, \ldots, z_m) \in \text{Conf}_m(\mathbb{C}P^1)/PSL(2, \mathbb{C})\}.
\]
Gromov compactification and stable maps

Just as for $\overline{M}_{g,m}$, the moduli space of stable maps $\overline{M}_{g,m}(M, J; \beta)$ is made up of different strata. We now describe how the different strata fit together. For this purpose, we again describe the strata in terms of the dual graphs augmented by the additional datum of homology classes $\beta_v$ attached to the vertex $v$.

**Definition 9.5.7** We define the intersection pattern, denoted by $D$, of the stable map $u$ to be the quadruple

$$D = (T, \{g_v\}, \{\beta_v\}, o) : \{1, \ldots, m\} \to \Sigma \setminus \text{sing}(\Sigma)$$

such that

1. $2g_v + m_v \geq 3$ or $\beta_v \neq 0$;
2. $\beta = \sum_v \beta_v$;
3. $g := \sum_v g_v + b_1(T)$; and
4. for $\beta_v$, there exists a $J$-holomorphic curve $u : \Sigma \to M$ with $u_*([\Sigma]) = \beta_v$ with genus($\Sigma$) = $g_v$.

We denote this data by the quadruple $D = (T, \{g_v\}, \{\beta_v\}, o)$. We say that $D$ and $D'$ are isomorphic if there exists an isomorphism $\phi : (T, \{g_v\}, \{\beta_v\}, o) \to (T', \{g'_v\}, \{\beta'_v\}, o')$ that satisfies $\beta'_{\phi(v)} = \beta_v$. We call an isomorphism class of $D$ an intersection pattern of the stable maps.

**Definition 9.5.8** Denote by $\text{Comb}(g, m; \beta)$ the set of isomorphism classes of $D = (T, \{g_v\}, \{\beta_v\}, o)$. We say that $D$ and $D$ are isomorphic if there exist finitely many vertices $v_1, \ldots, v_a$ of $T$ with decorations $g_{v_j}, \beta_{v_j}$ such that each vertex $v_j$ is replaced by an element

$$(T_{v_j}, (g_{v_j,w}), (\beta_{v_j,w}), o_{v_j}) \in \text{Comb}(g_{v_j}, m_{v_j}; \beta_{v_j}).$$

We denote by $M^D_{g,m}(M, J; \beta)$ the set of stable maps (or rather equivalence classes of them) whose intersection pattern is $D$.

**Definition 9.5.9** We denote by $\text{Sym}(D) \subset S_{\#V_T}$ the set of permutations $\sigma$ of $V_T$ such that $D$ and $D_{\sigma} = (T, \{g_{\sigma(v)}\}, \sigma \circ o)$ are isomorphic, and call it the symmetry group of $D$.

**Theorem 9.5.10** Let $g$, $m$ and $\beta$ be fixed, then

$$\#\text{Comb}(g, m; \beta) < \infty.$$
9.5 Stable map and stable map topology

**Proof** We start by stating the following lemma.

**Lemma 9.5.11** For any given non-negative integer \( g > 0 \), define

\[
A(\omega, J, M; \leq g) = \inf \{ \omega(u) \mid \partial_{j, j} u = 0, \ u : (\Sigma, j) \to (M, J) \text{ is non-constant and } \text{genus}(\Sigma) \leq g \}.
\]

Then \( A(\omega, J, M; \leq g) > 0 \).

**Proof** We prove this by induction on \( g \). When \( g = 0 \), the complex structure on \( \Sigma = S^2 \) is unique and hence we need only consider the map \( u \). In this case, the lemma easily follows from the \( \epsilon \)-regularity lemma. Suppose to the contrary that \( A(\omega, J, M; g = 0) = 0 \). Then there exists a sequence of non-constant \( J \)-holomorphic maps \( u_i : S^2 \to M \) such that \( \omega(u_i) \to 0 \). By the identity

\[
\frac{1}{2} \int_{\Sigma} |du|^2_J = \int_{\Sigma} u^* \omega, \tag{9.5.30}
\]

\( 0 < \int_{\Sigma} |du|^2_J \to 0 \). We equip \( S^2 \) with the standard metric and decompose \( S^2 \) into the union \( D_+ \cup D_- \) of discs. Obviously we have \( \int_{D_{\pm}} |du|^2_J < \epsilon_0 \) for all sufficiently large \( i \). Therefore we can apply the \( \epsilon \)-regularity theorem and the boot-strap argument to extract a \( C^1 \)-convergent subsequence of \( u_i \) on both \( D_{\pm} \) and hence on \( S^2 \). Denote the limit by \( u_\infty \). Then \( u_\infty \) is \( J \)-holomorphic, and \( u_\infty \) must have zero energy since the energy is continuous with respect to the \( C^1 \) topology. Therefore (9.5.30) implies that \( u_\infty \) must be the constant map and hence \( u_i \) converges to a constant map in \( C^1 \) topology. In particular, the \( u_i \) are homologous to the constant map and hence \( \omega(u_i) = 0 \) for sufficiently large \( i \). Again, by (9.5.30) we obtain \( \int_{\Sigma} |du|^2_J = 0 \), which contradicts the hypothesis that the \( u_i \) are all chosen to be non-constant.

Now suppose \( A(\omega, J, M; g) > 0 \) for all \( g < g_0 \) with \( g_0 \geq 1 \). By definition, we have

\[
A(\omega, J, M; \leq g_0) = \min_{0 \leq g \leq g_0} A(\omega, J, M; g).
\]

We would like to prove that \( A(\omega, J, M; g_0) > 0 \). Suppose to the contrary that there exists a sequence \( (\Sigma_i, j_i, u_i) \) with genus \( \Sigma_i = g_0 \) and \( \omega(u_i) \to 0 \). We add a sequence \( z_i \) of \( N \) marked points for a sufficiently large \( N \in \mathbb{N} \), and choose a subsequence of \( (\Sigma_i, j_i, z_i) \) so that \( (\Sigma_i, j_i, z_i) \to (\Sigma_\infty, j_\infty, z_\infty) \) in \( \overline{M}_{g_0, N} \). Now we consider the map \( u_i : (\Sigma_i, j_i) \to (M, J) \). If \( \Sigma_\infty \) remains smooth, then we can decompose \( \Sigma_\infty \) into a finite union \( \Sigma_\infty = \bigcup_{j=1}^K D_{\infty, j} \) of discs of radius \( r_0 > 0 \) with respect to a fixed metric in the conformal class of \( j_i \). Using the convergence of \( (\Sigma_i, j_i) \to (\Sigma_\infty, j_\infty) \), we can obtain the corresponding decomposition.
\[
\Sigma_i = \bigcup_{a=1}^{K} D_{i,a}, \quad D_{i,a} \to D_{\infty,a},
\]
where the convergence \( D_{i,a} \to D_{\infty,a} \) is \( C^\infty \). Since \( E_J(u_i) = \omega(u_i) \to 0 \), the \( \epsilon \)-regularity applied to each disc of the decomposition again, it follows that the \( u_i \) are homologous to a constant map. Then we obtain \( 0 = \omega(u_i) = E_J(u_i) > 0 \), which is a contradiction.

Therefore \( \Sigma_\infty \) cannot be smooth and hence has the decomposition

\[
((\Sigma_\infty, j_\infty), z_\infty) = \bigcup_{a=1}^{L} ((\Sigma_{\infty,a}, j_{\infty,a}), z_{\infty,a})
\]

with \( L > 1 \) into the irreducible components. We write

\[
((\Sigma_i, j_i), z_i) = \text{Res}_{((\Sigma'_i, j'_i), z'_i)}(\alpha_i),
\]

where \( ((\Sigma'_i, j'_i), z'_i) \) lie in the same stratum as that of \( ((\Sigma_\infty, j_\infty), z_\infty) \) and converge to \( ((\Sigma_\infty, j_\infty), z_\infty) \) in the stratum. We note that all the irreducible components of \( \Sigma_\infty \) have genus \( \leq g_0 \) and so do those of \( \Sigma'_i \).

Using this, we can decompose \( \Sigma_i \) into the union of \( W_i \cup (\Sigma_i \setminus W_i) \), where \( W_i = \bigcup_{e,a} W_{i,e,a} \) is the union of the annular regions

\[
W_{i,e,a} \cong [-L_{i,e,a}, L_{i,e,a}] \times S^1
\]

with flat metrics and \( L_{i,e,a} \to \infty \). Furthermore, \( W_{i,a} \) shrinks to the nodal point \( x_{e,a} \) with \( e = [v,w] \) and \( \Sigma_{i,a} \setminus W_i \to \Sigma_{\infty,a} \) and hence \( \Sigma_{i,a} \setminus W_i \) is a compact surface of genus \( \leq g_0 \) with a finite number of discs removed. Therefore we can decompose it into a fixed number \( N_0 \), which is independent of \( i \), of discs conformal to the flat disc of radius \( r_0 > 0 \).

Since \( E_J(u_i) \to 0 \) and \( j_{i,a} \to j_{\infty,a} \) on \( \Sigma_{i,a} \setminus W_i \), we can apply the \( \epsilon_0 \)-regularity to each such disc of \( \Sigma_{i,a} \setminus W_i \) and to the \( W_{i,e,a} \) (which has bounded curvature and injective radius bounded away from 0), and thus we conclude that all of them converge to constant maps. Since the \( \Sigma_i \) are connected, this implies that all the maps \( u_i \) are homologous to constant maps. In particular, we must have \( \omega(u_i) = 0 \), which is a contradiction to the hypothesis that the \( u_i \) are non-constant.

Let \( \delta = A(\omega, J, M; \leq g) \), and set \( N = \lfloor \omega(\beta)/\delta \rfloor \). Denote

\[
V = \#(V_T), \quad S = \sum_{v \in V_T} g_v.
\]

We observe that the following bounds hold:

\begin{enumerate}
\item \( \# \{ v \mid g_v \geq 1 \} \leq g \)
\item \( \# \{ v \mid \text{mark}(v) > 0 \} \leq m \)
\end{enumerate}
Let $V_0$ be the sum of the cardinals on the left-hand sides of (1)–(4). Then we have

$$V_0 \leq 2g + m + N. \quad (9.5.31)$$

For any other vertex $v$, we must have $g_v = 0$, $\text{mark}(v) = 0$ and $\beta_v = 0$. By the stability condition, we must have $\text{sing}(v) \geq 3$ for such $v$. However, we have shown before in the proof of Theorem 9.3.14 (see (9.3.7) and (9.3.10)) that

$$\# \{ v \mid \text{sing}(v) \geq 3 \} \leq \# \{ v \mid \text{sing}(v) = 1 \} - 2\chi(T)$$

$$\quad = \# \{ v \mid \text{sing}(v) = 1 \} - 2S + 2g - 2. \quad (9.5.32)$$

Furthermore, by the stability condition, we have

$$\# \{ v \mid \text{sing}(v) = 1 \} \leq \# \{ v \mid g_v > 0 \} + \# \{ v \mid g_v = 0, \text{mark}(v) = 0, \beta_v \neq 0 \}$$

$$+ \# \{ v \mid \text{mark}(v) \geq 1 \}$$

$$\leq g + N + m. \quad (9.5.33)$$

Therefore, from (9.5.31)–(9.5.33), we derive

$$V \leq V_0 + \# \{ v \mid \text{sing}(v) \geq 3 \}$$

$$\leq 2g + m + N + (g + N + m) - 2S + 2g - 2$$

$$\leq 5g + 2m + 2N - 2S - 2 \leq 3g + 2m + 2N - 2.$$

This finishes the proof. \[ \square \]

The following proposition provides a concrete meaning of the stability condition on the pair $((\Sigma, z), u)$.

**Proposition 9.5.12** The nodal $J$-holomorphic map $((\Sigma, z), u)$ is stable if and only if, for any irreducible component $\Sigma_v$, one of the following alternatives holds:

1. $u \circ \pi_v : \Sigma_v \to M$ is not constant or
2. $m_v + 2g_v \geq 3$, i.e., the domain $(\Sigma_v, z_v)$ is stable.

**Proof** It is obvious that, if a $J$-holomorphic map $u$ is stable and constant, then the domains of all the irreducible components $((\Sigma_v, z_v), u_v)$ must be stable and hence (2) must apply. Here we denote $u_v = u \circ \pi_v : \Sigma_v \to M$ and by $z_v$ the union

$$z_v = \pi_v^{-1}(z) \cup \text{sing}(\Sigma_v),$$

and $m_v = \text{mark}(v) + \text{sing}(v)$. Therefore we need only prove the ‘if’ part.
If the domain \((\Sigma_v, z_v)\) is stable, i.e., satisfies (2), then obviously we have 
\[ \#\text{Aut}((\Sigma_v, z_v), u_v) < \infty. \]

Now suppose \(u_v = u \circ \pi_v\) is not constant, i.e., satisfies (1). Then McDuff’s structure theorem, Theorem 8.7.3, implies that there exists a Riemann surface \(\Sigma'_v\), a branched covering \(\phi : \Sigma_v \to \Sigma'_v\) and a \(J\)-holomorphic map \(u'_v : \Sigma'_v \to M\) such that \(u_v = u'_v \circ \phi\), where \(u'_v\) is somewhere injective.

Let \(z'_0 \in \Sigma'_v\) be a point such that \(du'_v(z'_0) \neq 0\), with \((u'_v)^{-1}(z'_0) = \{z'_0\}\). Then we can find a neighborhood \(V \subset \Sigma'\) of \(z'_0\) such that
\[
(u'_v)^{-1}(z') = \{z'\}, \quad du'_v(z) \neq 0
\] (9.5.34)
for all \(z' \in V\). Therefore \(\phi^{-1}(V) \to V\) is a covering map and \(\phi^{-1}(V) = \bigcup_{i=1}^k U_i\) for some \(k \in \mathbb{N}\), where each \(U_i\) is a connected component of \(\phi^{-1}(V)\) that is isomorphic to \(V\).

Now suppose to the contrary that \(\#\text{Aut}((\Sigma_v, z_v), u_v) = \infty\) and hence there exists an infinite sequence \(\psi_i \in \text{Aut}((\Sigma_v, z_v), u_v)\) such that \(\psi_i \neq \psi_j\) for \(i \neq j\). By definition, we have
\[
\psi_i \in \text{Aut}(\Sigma, z), \quad u_v \circ \psi_i = u_v.
\]
Hence we have \(u'_v \circ \phi \circ \psi_i = u'_v \circ \phi\) from \(u_v = u'_v \circ \phi\). Because of the somewhere injectivity (9.5.34) of \(u'_v\), this implies that \(\phi \circ \psi_i|_{\phi^{-1}(V)} = \phi|_{\phi^{-1}(V)}\). In particular, the automorphisms \(\psi_i\) map \(\phi^{-1}(V)\) to \(\phi^{-1}(V)\) for all \(i\). Since there are only finitely many sheets in \(\phi^{-1}(V)\) and the \(\psi_i\) are continuous, there exist two connected components \(U_{i_0}, U_{i_1} \subset \phi^{-1}(V)\) such that we can select a subsequence \(\psi_{i_l}\) that maps \(U_{i_0}\) to \(U_{i_1}\) for all \(l\). Then \(\psi_{i_l}|_{U_{i_0}} \equiv \psi_{i_k}|_{U_{i_0}}\) follows from \(\phi \circ \psi_{i_l}|_{\phi^{-1}(V)} = \phi \circ \psi_{i_k}|_{\phi^{-1}(V)}\) for all \(l, k\). By the unique continuation, this implies that \(\psi_{i_l} \equiv \psi_{i_k}\) on \(\Sigma\), which contradicts the hypothesis \(\psi_i \neq \psi_j\) for \(i \neq j\).

Finally we recall that \(\text{Aut}((\Sigma, z), u)\) is a subset of the product of the symmetry group \(\text{Sym}(D)\) of the dual graph \(D\) and the fiber product
\[
\prod_{v \in V_T^u} \text{Aut}((\Sigma_v, z_v), u_v) \times \prod_{v \in V_T^u} \text{Aut}((\Sigma_v, z_v), u_v)
\]
under the matching conditions \(\phi_e(x_v) = \phi_w(x_w)\) for each element \(e = [v, w] \in E_T\). Here \(V_T^u\) and \(V_T^{uu}\) are the sets of vertices consisting of stable and unstable ones respectively.

Obviously we have that \(\#\text{Aut}(D) < \infty\). Since we have shown that all \(\text{Aut}((\Sigma_v, z_v), u_v)\) are finite groups, \(\text{Aut}((\Sigma, z), u)\) must be finite. This finishes the proof. \(\square\)
It follows from Proposition 9.5.12 that condition (1) implies that any nonconstant $J$-holomorphic curve is stable. We form the union

$$\overline{\mathcal{M}}_{g,m}(M, J; \beta) = \bigcup_{D \in \text{Comb}(g,m,\beta)} \mathcal{M}_{g,m}^D(M, J; \beta).$$

**Remark 9.5.13** The space $\overline{\mathcal{M}}_{g,m}(M, J)$ has a canonical filtration induced by the symplectic area. Let $K > 0$ and denote

$$\overline{\mathcal{M}}_{g,m}(M, \omega, J; \omega(\beta) \leq K) = \bigcup_{\omega(\beta) \leq K} \overline{\mathcal{M}}_{g,m}(M, \omega, J; \beta),$$

$$\overline{\mathcal{M}}_{g,m}(M, \omega, J) = \bigcup_{K \geq 0} \overline{\mathcal{M}}_{g,m}(M, J; \omega(\beta) \leq K).$$

We cannot expect $\overline{\mathcal{M}}_{g,m}(M, \omega, J)$ to be compact, since there is a sequence of holomorphic maps whose areas go to infinity. But we will show that $\overline{\mathcal{M}}_{g,m}(M, \omega, J; \omega(\beta) \leq K)$ is compact for all $K \geq 0$ later, which is basically the content of the precise version of the Gromov compactness theorem (Gr85).

**9.5.2 Definition of stable map topology**

We will now equip $\overline{\mathcal{M}}_{g,m}(M, J)$ with the direct limit topology of $\overline{\mathcal{M}}_{g,m}(M, \omega, J; \omega(\beta) \leq K)$ as $K \to \infty$. This topology will be Hausdorff and its subspace topology on $\overline{\mathcal{M}}_{g,m}(M, \omega, J; \omega(\beta) \leq K)$ will be compact.

A crucial difficulty in providing such a topology lies in the facts that the domains of stable maps are not necessarily stable and that we allow the conformal structures (or complex structures) to degenerate. Because the space $\overline{\mathcal{M}}_{g,m}(M, J; \beta)$ is not the set of maps but that of equivalence classes of $(u, j)$ maps $u : (\Sigma, z) \to M$ with a complex structure $j$ specified, one has to simultaneously deal with this degeneration together with the normalization of unstable domains.

To illustrate the subtlety of this convergence in the general case, we examine one example from (FOn99).

**Example 9.5.14** Consider the union of two rational irreducible curves, $C_1$ and $C_2$ in $\mathbb{CP}^2$, of degree 1 and 2, respectively. In fact, it follows from a simple intersection argument that they must be smooth and embedded and that they intersect at two distinct points. The homology class is $\beta_0 = 3[H]$. 


Consider the domain \((\Sigma, z) = (\Sigma_1, z_1) \cup (\Sigma_2, z_2)\) as above, which is stable and ‘rigid’. This stable curve has genus 1. We denote by \(D_0\) this rigid intersection pattern of \((\Sigma_a, z_a)\) for \(a = 1, 2\). See Figure 9.9.

Consider any sequence \(u_i : (\Sigma, z) \rightarrow \mathbb{CP}^2\) such that the two nodal image points \(x_i, y_i \in \mathbb{CP}^2\) approach each other. Denote

\[
u_{i,a}^{-1}(x_i) = s_{i,a} \in \Sigma_a,
\]

\[
u_{i,a}^{-1}(y_i) = t_{i,a} \in \Sigma_a
\]

for \(a = 1, 2\). By applying \(PSL(2, \mathbb{C})\) on \(\Sigma_a\), we may assume that \(s_{i,a} = s_a\) and \(t_{i,a} = t_a\) are fixed over \(i\) under a fixed identification of \(\Sigma_a \cong \mathbb{CP}^1\). The components \(u_{i,a} : \Sigma_a \rightarrow \mathbb{CP}^2\) must satisfy

\[
\text{dist}(u_{i,a}(s_a), u_{i,a}(t_a)) \rightarrow 0, \quad a = 1, 2,
\]

since

\[
u_{i,a}(s_a) = x_i, \quad \nu_{i,a}(t_a) = y_i.
\]

The following is a basic question to ask.

**Question 9.5.15** Can we find a convergent sequence of \(u_i\) in \(C^\infty\) topology whose domain \((\Sigma, z)\) keeping the intersection pattern \(D_0\) fixed, but the limit of their image curves is the nodal curve with a singular point as in Figure 9.10?

We claim that this is impossible for the following reasons.

1. If a sequence \(u_i\) with such a domain converges to a limit map \(u_\infty = (u_{1,\infty}, u_{2,\infty})\) whose domain remains in \(\overline{M}_{1,2}\), then both maps \(u_{a,\infty}\) for \(a = 1, 2\) must have singular values at \(s_a, t_a\).

2. Since \(\Sigma_a (a = 1, 2)\) are rigid, the limit curve cannot be reducible, since irreducible curves in these homology classes are embedded and hence the limit must be embedded too. In particular, their ramification orders must remain the same in the limit. This gives rise to a contradiction to statement (1).
9.5 Stable map and stable map topology

The lesson we have learned from this example is that the image of this particular stable map, say $\tau$, cannot be that of the limit of stable maps with domain $(\Sigma, z)$ of the fixed intersection pattern $D_0$, but must have the same image as another stable map with its domain of different intersection patterns.

Now comes what is really going on in this example. Consider $(u_i, (\Sigma, z))$, whose images resemble $C_{1,i} \cup C_{2,i}$. We want to describe the convergence $(u_i, (\Sigma, z)) \to (u_\infty, (\Sigma, z))$ in $\overline{M}_{1,2}(\mathbb{C}P^2, 3[H])$.

Note that in the above ill-made convergence argument the domains of $u_i : (\Sigma, z) \to \mathbb{C}P^2$ are stable, but the domain of the expected limit map is not.

To be able to define a correct notion of convergence, we need to add one more marked point to each component so that the maps $u_2, u_3$ converge to constant maps to a point in $C_1 \cap C_2$. Now, after removing added marked points, we get the desired limit. The limit has four irreducible components, rather than two, two of which are constant components with stable domains.

To handle the above-mentioned instability problems systematically, we use the method of adding extra marked points. This will be used both to stabilize the domains and to properly encode the bubbling phenomenon.

We first give the definition of sequential convergence and then provide a basis of the corresponding topology at the end.

I. Stable cases

Denote by $\overline{M}_{g,m}^{st}(M, J; \beta) \subset \overline{M}_{g,m}(M, J; \beta)$ the subset of (equivalence classes of) stable maps with stable domains. We denote by $\overline{M}_{g,m}(M, J; \beta)$ and $\overline{M}_{g,m}^{st}(M, J; \beta)$ the set of stable maps and of those with stable domains, not just equivalence classes. We will first define the meaning of sequential convergence on $\overline{M}_{g,m}^{st}(M, J; \beta)$ by using the sequential convergence for the set of
stable maps in $\overline{\mathcal{M}}_{g,m}^{st}(M, J; \beta)$. Finally, we will give the definition of the basis of the corresponding topology on $\overline{\mathcal{M}}_{g,m}^{st}(M, J; \beta)$.

Let $[\Sigma, z] \in \overline{\mathcal{M}}_{g,m}$ with the domain of the intersection pattern given by $D = (T, \{g_v\}, 0, \{\beta_v\})$. Denote $\Sigma = \cup_v \Sigma_v$ and consider a sequence of stable maps $((\Sigma_i, z_i), u_i)$. We also denote the union of neck regions by

$$\overline{\mathcal{M}}_{g,m}(M, J; \beta)$$

for a sequence $(\Sigma_i', z_i') \in \mathcal{M}_{g,m}^{D'}$, and for sufficiently small gluing parameters

$$(\alpha_{e,i})_{e \in E_T} \in \bigoplus_{e=[v,w]} T_{x_{ei}} \Sigma_{v,i} \otimes T_{x_{wi}} \Sigma_{w,i}$$

given at singular points so that

$$[\Sigma_i', z_i'] \to [\Sigma, z] \quad \text{in} \quad \mathcal{M}_{g,m}^{D} \quad \text{and} \quad (\alpha_{e,i}) \to 0$$
as $i \to \infty$.

We recall that we set $R_{e,i}^{-1/2} = |\alpha_{e,i}|$ and that $\Sigma_i$ has a subset identified with $\Sigma_i' \cup_{x_{ei}} D_{x_{ei}} (R_{e,i}^{-3/2})$ by definition. Also, in terms of the explicitly given metrics on $\text{Res}(\Sigma, z) (\{\alpha_e\}_{e \in E_T})$ as described in Example 9.4.21, we have

$$\text{diam} \left( \Sigma_i' \setminus \left( \Sigma_i' \cup_{x_{ei}} D \left( R_{e,i}^{-3/2} \right) \right) \right) \to 0 \quad \text{as} \quad i \to \infty.$$ 

For a given $\mu > 0$ and a collection of sufficiently large $R_{e,i}$, we denote by

$$W_{e,i}(\mu) := \left( D_{x_{ei}} (\mu) - D_{x_{ei}} (R_{e,i}^{-1}) \right) \cup \left( D_{x_{ei}} (\mu) - D_{x_{ei}} (R_{e,i}^{-1}) \right)$$

the prescribed neck region at $x_{ei} = (x_{vi}, x_{wi}) \in \text{Sing}(\Sigma_i)$, where

$$(\Sigma_i, z_i) = \text{Res}(\Sigma_i', z_i') (\{\alpha_{e,i}\}).$$

See Figure 9.11.

We also denote the union of neck regions by

$$W_i(\mu) = \bigcup_{e \in E_T} W_{e,i}(\mu).$$

**Definition 9.5.16** We denote $s\text{-lim}_{i \to \infty} ([\Sigma_i, z_i]; u_i) = ([\Sigma, z], u)$ if

1. $\lim_{i \to \infty} [\Sigma_i, z_i] = [\Sigma, z]$ in $\overline{\mathcal{M}}_{g,m}$
2. for any $\mu > 0$, $u_i|_{\Sigma_i', W_{\cdot,i}(\mu)} \to u$ in $C^\infty$ on compact sets
(3) \( \lim_{\mu \to 0} (\limsup_{i \to \infty} \text{diam}(u_i(W_{e,i}(\mu)))) = 0 \) at the singular point \( x_e \) for all \( e \in E_T \).

We note that the convergence stated in condition (1) in this definition means that there are representatives \((\Sigma_i, z_i)\) and \((\Sigma, z)\) of the corresponding equivalence classes such that \((\Sigma_i, z_i)\) converges to \((\Sigma, z)\) in the sense described in Proposition 9.3.19 and Theorem 9.4.27.

The above prescription can be turned into that of a neighborhood basis of a topology at infinity on \( \mathcal{M}_{g,m}^{st}(M; J; \beta) \) as follows. For the sake of notational simplicity, we also denote a stable curve by \( v = [\Sigma, z] \) and a stable map by \( x = [(\Sigma, z), u] \).

There exists a neighborhood \( U(\text{Sing}(\Sigma)) \) of the nodal point set \( \text{Sing}(\Sigma) \) of \( \Sigma \) such that we have a canonical smooth embedding

\[
i_v : \Sigma \setminus U(\text{Sing}(\Sigma)) \to \text{Res}(\Sigma, \{\alpha_e\}_{e \in E_T})
\]

such that \( i_v(z_i) = z_i \). We may assume that \( i_v \) is holomorphic outside \( U(\text{Sing}(\Sigma)) \). We often identify \( \Sigma \setminus U(\text{Sing}(\Sigma)) \) with its image in \( \text{Res}(\Sigma, \{\alpha_e\}_{e \in E_T}) \) via \( i_v \).

Equip \( \mathcal{M}_{g,m} \) with an appropriate metric and define a distance on it, recalling that \( \mathcal{M}_{g,m} \) is an orbifold (DM69) and can be given a Riemannian metric. We denote by \( \mathcal{U}(v) \) a neighborhood of \( v \) therein.

**Definition 9.5.17** For each given \( x \) and \( \epsilon > 0 \), we define the set consisting of the stable maps \( x' = (\Sigma', z', u') \) that satisfy the following conditions.
(1) $v' = (\Sigma', z') \in \U(v)$ and $\text{dist}(v', v) < \epsilon$.

(2) There exists a biholomorphism $\varphi$ such that the $C^1$ distance between the two maps $u' \circ i_{vv'}$ and $u \circ \varphi$ is smaller than $\epsilon$ on $\Sigma \setminus \U(\text{Sing}(\Sigma))$.

(3) $\text{diam}(u'(S)) < \epsilon$ if $S$ is any connected component of the complement $\Sigma' \setminus i_{vv'}(\Sigma \setminus \U(\text{Sing}(\Sigma)))$.

Then we have the following lemma.

**Lemma 9.5.18** The collection $\U^*(\epsilon, x)$ consisting of the isomorphism classes of the stable maps described in Definition 9.5.17 defines a basis of a topology on $\overline{M}_{g,m}(M, J; \beta)$.

**Exercise 9.5.19** Prove this lemma.

### II. General cases

In general, the domain $(\Sigma, z)$ of a stable map need not be stable and hence might not define an element in $\overline{M}_{g,m}$. To define a convergence, we carry out the following steps for a given sequence $[(\Sigma_i, z_i), u_i]$.

1. Add additional marked points $w_i = \{w_{i,a} \mid a \in A\}$ for some indexing set $A$ with $\#A < \infty$, so that $(\Sigma_i, z_i \cup w_i)$ becomes stable.
2. Take the limit of $[(\Sigma_i, z_i \cup w_i), u_i]$ in the sense of s-lim$_{i \to \infty}$.
3. After taking the limit, we remove the limit of added marked points.

**Definition 9.5.20** Let $[(\Sigma_i, z_i), u_i], [(\Sigma, z), u] \in \overline{M}_{g,m}(M, J; \beta)$. We say that

$$\lim_{i \to \infty} [(\Sigma_i, z_i), u_i] = [(\Sigma, z), u] \text{ in } \overline{M}_{g,m}(M, J; \beta)$$

if the following holds. There exist stable curves $(\Sigma_i, z_i^+)$ with $z_i^+ = z_i \cup w_i$ with $\#w_i = n$ and $(\Sigma_\infty, z_\infty^+)$ and a stable map $((\Sigma_\infty, z_\infty^+), u_\infty) \in \overline{M}_{g,m+n}(M, J; \beta)$ such that

1. s-lim$_{i \to \infty} [(\Sigma_i, z_i^+), u_i] = [(\Sigma_\infty, z_\infty^+), u_\infty]$ with $z_\infty^+ \text{ in } \overline{M}_{g,m+n}(M, J; \beta)$ and
2. we have

$$\text{forget}_{m+n, n}[(\Sigma_\infty, z_\infty^+), u_\infty] = [(\Sigma, z), u].$$

The forgetful map $\text{forget}_{m+n, n} : \overline{M}_{g,m+n}(M, J; \beta) \to \overline{M}_{g,m}(M, J; \beta)$ induces the process of contracting the unstable component with a constant map to a point to make sure that the resulting curve is stable.

Now let us describe the topology associated with the above-mentioned sequential convergence on $\overline{M}_{g,m}(M, J; \beta)$. We emphasize that $(\Sigma, z)$ might not be stable. We choose a finite number of additional marked points
on $\Sigma$ for some finite index set $A$. We denote $l = \#A$. To make this choice turn into a continuous choice near a given representative $((\Sigma, z), u)$ of $x \in \overline{M}_{g,m}(M, J; \beta)$, we require the following.

**Conditions 9.5.21**

(1) $w$ is disjoint from $z$ and the set of nodal points of $\Sigma$.

(2) $(\Sigma, w)$ is stable. Moreover, it has no non-trivial automorphism.

(3) $u$ is immersed at $w_a$.

It is always possible to choose such additional points from $\Sigma$. We consider $v = [\Sigma, z \cup w] \in \overline{M}_{g,m+l}$, and take neighborhoods $U(v)$ of $v$ in $\overline{M}_{g,m+l}$ with the following properties. There exists a neighborhood $U(S(\Sigma))$ of the singular point set $S(\Sigma) = \text{Sing}(\Sigma)$ of $\Sigma$ such that for any $v' = [\Sigma', z' \cup w'] \in U(v)$ we have a smooth embedding

$$i_{vv'} : \Sigma \setminus U(S(\Sigma)) \to \Sigma'$$

such that

$$i_{vv'}(z_i) = z'_i, \quad i_{vv'}(w_a) = w_a.$$

We may assume that $i_{vv'}$ is holomorphic outside $U(S(\Sigma))$.

We also require that $i_{vv'}$ depends continuously on $v'$ in the $C^\infty$ sense.

Now, for a given $x = [(\Sigma, z), u]$, we fix $v = [\Sigma, z \cup w]$. Since the choice of $w$ depends on the representative $\tilde{x} = ((\Sigma, z), u)$ of $x \in \overline{M}_{g,m}(M, J; \beta)$, we denote $w = w(\tilde{x})$.

For each given $w_a, a \in A$, we choose a normal slice $N_{w_a}^\prime$ of $u$ such that the following conditions apply.

**Conditions 9.5.22** (Normal slice)

(1) $N_{w_a}$ is a smooth submanifold of codimension 2.

(2) $N_{w_a}$ intersects transversally with $u(\Sigma)$ at $u(w_a)$.

We remark that the choices of the above slices $N_{w_a}$ depend only on $\tilde{x} = ((\Sigma, \tilde{z}), u)$ and $w = \{w_a\}_{a \in A}$, which will be fixed once and for all when the latter is given. The slices also carry the metrics induced from the one given in the ambient space $M$.

Equip an appropriate metric with $\overline{M}_{g,\ell+m}$ to define the distance between $v'$ and $v$. 
Definition 9.5.23 (Stabilization)  For each given \( x \) and its representative \( \overline{x} = (\Sigma, z, u) \) and \( \epsilon_1, \epsilon_2 > 0 \), we define the set \( \mathcal{V}(\epsilon_1, \epsilon_2, \overline{x}) \) consisting of the stable maps \( \overline{x}' = ((\Sigma', z'), u') \) that satisfy the following conditions.

1. \( v' = [\Sigma', z' \cup w'] \in \mathcal{U}(v) \) and \( \text{dist}(v', v) < \epsilon_1 \).
2. There exists a biholomorphism \( \varphi \) such that the \( C^1 \) distance between the two maps \( u' \circ i_{v'} \) and \( u \circ \varphi \) is smaller than \( \epsilon_1 \) on \( \Sigma \setminus \mathcal{U}(S(\Sigma)) \).
3. \( \text{diam}(u'(S)) < \epsilon_1 \) if \( S \) is any connected component of the complement \( \Sigma' \setminus i_{v'}(\Sigma \setminus \mathcal{U}(S(\Sigma))) \).
4. Denote \( w' = (w'_a)_{a \in A} \). We require \( u'(w'_a) \in N_{w_a} \) for each \( a \) and

\[
\text{dist}(u'(w'_a), u(\varphi(w_a))) < \epsilon_2.
\]

When \( w \) and \( w' \) are specified as in the above condition, we say that \( \overline{x}' = ((\Sigma', z'), u') \) is \((\epsilon_1, \epsilon_2)\)-close to \( \overline{x} \) with respect to \( w \).

Here we would like to emphasize that \( w' = (w'_a) \) are uniquely determined by (4) by the choices of \( w \) and \( N_{w_a} \), provided that \( \epsilon_1, \epsilon_2 \) are sufficiently small, since \( N_{w_a} \) are transversal to \( u \). On the one hand, we introduce a finite number of extra marked points \( w' = \{w'_a\} \), and also introduce the same number of codimension-2 requirements

\[
u'(w'_a) \in N_{w_a}, \quad \tag{9.5.38}\]

Therefore the resulting (virtual) dimension of the set of \( \{(\Sigma', z' \cup w'), u'\} \) is the same as that of the original moduli space \( \overline{\mathcal{M}}_{g,m}(M, J; \beta) \).

Finally we define a subset of \( \overline{\mathcal{M}}_{g,m}(M, J; \beta) \) by

\[
\mathcal{U}(x; \epsilon_1, \epsilon_2) = \{((\Sigma', z'), u') \in \overline{\mathcal{M}}_{g,m}(M, J; \beta) \mid ((\Sigma', z'), u') \in \mathcal{V}(\epsilon_1, \epsilon_2, \overline{x})\},
\]

which is the set of equivalence classes of the elements coming from \( \mathcal{V}(\epsilon_1, \epsilon_2, \overline{x}) \). If we denote by

\[
\mathcal{V}^{st}(\epsilon_1, \epsilon_2, \overline{x}; w(\overline{x}))
\]

the set

\[
\{((\Sigma', z' \cup w'), u') \mid ((\Sigma', z'), u') \in \mathcal{V}(\epsilon_1, \epsilon_2, \overline{x})\}
\]

and by

\[
\mathcal{U}^{st}(\epsilon_1, \epsilon_2, \overline{x}; w(\overline{x})) \subset \overline{\mathcal{M}}^{st}_{g,m+n}(M, J; \beta)
\]

its projection, then we can also express

\[
\mathcal{U}(x; \epsilon_1, \epsilon_2) = \text{forget}_{m+n,m}(\mathcal{U}^{st}(\epsilon_1, \epsilon_2; \overline{x}, w(\overline{x}))) \quad \tag{9.5.40}
\]
by construction. We note that, by the requirement that \((\Sigma, w(\tilde{x}))\) has a trivial automorphism, \(V^{st}(\epsilon_1, \epsilon_2, \tilde{x}; w(\tilde{x}))\) projects to \(\overline{\mathcal{M}}_{g,m+n}(M; J; \beta)\) injectively.

The main proposition then is the following.

**Proposition 9.5.24** The collection \(U(x; \epsilon_1, \epsilon_2)\) over all \(x, \epsilon_1, \epsilon_2 > 0\) provides a basis of a topology on \(\overline{\mathcal{M}}_{g,m}(M, J; \beta)\). Furthermore the topology does not depend on the choices of \(w\) and the \(N_{\psi_a}\) which enter into the construction of \(U(x; \epsilon_1, \epsilon_2)\).

**Exercise 9.5.25** Prove this proposition and that the topology is equivalent to the topology induced by the sequential convergence described above.

### 9.5.3 Compactness

In this section, we will prove the compactness of the stable map topology.

**Theorem 9.5.26** \(\overline{\mathcal{M}}_{g,m}(M, \omega, J; \beta)\) is compact.

The proof of this compactness will be divided into several steps according to the definition of the topology described by its basis element

\[ U(x; \epsilon_1, \epsilon_2) = \text{forget}_{m+n}(U^{st}(\epsilon_1, \epsilon_2; \tilde{x}, w(\tilde{x}))) \]

given in (9.5.40) or according to the definition of sequential convergence given in Definition 9.5.20.

We also remark that there are three sources of the failure of convergence in the space of stable maps in \(\overline{\mathcal{M}}_{g,m}(M, J; \beta)\).

1. The map goes to constant on some unstable components.
2. Two or more marked points collide.
3. The derivative blows up, i.e., \(|du_i(z_i)| \to \infty\) for some sequence \(z_i\).

### Initial stabilization of domains

Let \(((\Sigma_i, z_i), u_i)\) be a given sequence of stable maps. Using the finiteness of combinatorial types of \(\overline{\mathcal{M}}_{g,m}(M, J; \beta)\), we may assume that \([((\Sigma_i, z_i), u_i)] \in \mathcal{M}_{g,m}(M, J; \beta)\) for a fixed \(D\), after choosing a subsequence. Therefore, we can add the same finite number \(n\) of additional marked points \(w_i = \{w_i, a\}_{a \in A}\) with \(\#(A) = n\) to \((\Sigma_i, z_i)\) and obtain a stable curve \((\Sigma_i, z_i \cup w_i)\). We denote \(z_i^+ = z_i \cup w_i\).

Using the compactness of \(\mathcal{M}_{g,m+n}\), after choosing a subsequence, we may assume that

\[
\lim_{i \to \infty}(\Sigma_i, z_i^+) = (\Sigma_{\infty}^+, z_{\infty}^+) \quad (9.5.41)
\]
in \( \overline{\mathcal{M}}_{g,m+n} \). Denote by
\[
D^+ = (T^+, \{g^+_v\}, o^+)
\]
the intersection pattern of \((\Sigma^+_{\infty}, z^+_{\infty})\).

Decompose \(\Sigma^+_{\infty} = \bigcup_v \Sigma^+_{\infty,v}\) into irreducible components. Using the definition of the convergence in (9.5.41), we can write
\[
(\Sigma_i, z^+_i) = \text{Res}(\Sigma'_i, z'^+_i)((\alpha_{e,i})), \quad \lim_{i \to \infty} \alpha_{e,i} = 0,
\]
for some \((\Sigma'_i, z'^+_i)\) with the same intersection pattern \(D^+\) and with deformation parameters \((\alpha_{e,i}, i_{x,v^i}) \in T_{x,v^i} \Sigma^+_{\infty,v^i} \otimes T_{x,v^i} \Sigma^+_{\infty,v^i}\) such that
\[
\lim_{i \to \infty}(\Sigma'_i, z'^+_i) = (\Sigma^+_{\infty}, z^+_{\infty}) \text{ in } \mathcal{M}^{D^+}_{g,m+n}. \tag{9.5.42}
\]
By virtue of the convergence (9.5.42), we can choose \(\epsilon_0 > 0\) independent of \(i\) but depending only on \((\Sigma^+_{\infty}, z^+_{\infty})\), and the complex coordinates \(z\) given on \(U_{x_v} \subset \Sigma^+_{\infty,v}\), a neighborhood of a singular point in \(\Sigma^+_{\infty,v}\) such that
\[
z(U_{x_v}) \supset D^2(\epsilon_0)
\]
for all \(v \in V_T\) and \(x_v \in \text{Sing}(\Sigma^+_{\infty,v})\).

**Remark 9.5.27** What is really going on in this step of stabilizing the domains and taking their limits is that we are extracting a sequence of metrics on the domains \(\Sigma_i, \Sigma'_i\) which converge to the one given in \(\Sigma^+_{\infty}\) as described in Fukaya–Ono deformation, especially as prescribed near the neck regions in the deformation. This enables one to study the \(C^1\)-convergence of the maps \(u_i\) on the stabilized domains on any given compact subset of \(\Sigma^+_{\infty}\) away from the singular points thereof.

**Convergence away from neck regions**

In terms of Fukaya–Ono deformation, we can write \(\Sigma_i = \text{Res}(\Sigma'_i, z'^+_i)((\alpha^i_{x,v^i}))\), which has a subset, the image of \(i_{v^i,v^i}\), identified with
\[
\Sigma'_i - \bigcup_{x,v^i} D_{e,i}(R_{e,i}^{-3/2}) \subset \Sigma'_i.
\]
Here we denote
\[
v'^+_i = (\Sigma'_i, z'^+_i), \quad v_i = (\Sigma_i, z^+_i)
\]
with \(R_{e,i} = |\alpha^i_{x,v^i}|^{-1/2}\).

**Proposition 9.5.28** Let \(\mu > 0\) be any given small constant with \(\mu < \epsilon_0\). On adding more marked points to those \(w_i\) chosen in the initial stabilization and
9.5 Stable map and stable map topology

Figure 9.12 A thin neck.

taking a subsequence, still denoted by $z_i^+ = z_i \cup w_i$, there exists a constant $C > 0$ such that

$$\sup_{\Sigma_i^+ \cup \bigcup_{i=1}^n D(u_i, \mu)} |du_i| \leq C,$$

where $C$ is independent of $i$ (but may depend on $\mu$).

Before carrying out the proof of Proposition 9.5.28, we recall several lemmata that will be used in the proof. The proof of the following decay estimate will be given in Appendix B.

Lemma 9.5.29 There exist $L_0 > 0$ and $\epsilon_1 > 0$ such that for any $L \geq L_0$, if the map $u : [-L - 1, L + 1] \times S^1 \to (M, J)$ is $J$-holomorphic and $\text{diam}(u([-L - 1, L + 1] \times S^1)) \leq \epsilon_1$, then

$$|\frac{\partial u}{\partial \tau}(\tau, t)| + |\frac{\partial u}{\partial t}(\tau, t)| \leq Ce^{-\lambda \text{dist}(\tau, \partial[-L-1,L+1])}$$

for $\tau \in [-L, L] \times S^1$, where $\lambda > 0$, $C > 0$ is independent of $L \geq L_0$.

This lemma roughly says that, when the diameter of the image of a $J$-holomorphic map defined on a long cylinder is sufficiently small, then its image looks like a hyperbolic surface as in Figure 9.12.

An immediate corollary of this exponential estimate is the following derivative estimate on $J$-holomorphic maps defined on the annuli in $\mathbb{C}$.

Corollary 9.5.30 Consider the conformal identification

$$[-L, L] \times S^1 \simeq \text{Ann}(r, 1) : w \mapsto e^{-L}e^w,$$
where \( r = e^{-2L} \) and \( z \) the standard complex coordinate on \( D^2(1) \subset \mathbb{C} \). Then we have

\[
\left| \frac{\partial u}{\partial z} (z) \right| + \left| \frac{\partial u}{\partial \bar{z}} (z) \right| \leq C \lambda \max \left( 1, \frac{r}{|z|^2} \right)
\]

for all \( z \) with \( \log |z| \in [\log r + 1, \log r - 1] \). (\( z \) is in a slightly smaller domain of the annulus.)

**Proof of Proposition 9.5.28** Let \( \epsilon_1 \) be the constant given in Lemma 9.5.29. Let \( \mu > 0 \) be given. By taking a subsequence, we may assume that, for each component \( v \), either

\[
\text{diam} \ u_i(\Sigma'_{i,v} - \bigcup_{x_{i,j}} D_{x_{i,j}}(\mu)) \geq \frac{\epsilon_1}{1000} \quad (9.5.43)
\]

or

\[
\text{diam} \ u_i(\Sigma'_{i,v} - \bigcup_{x_{i,j}} D_{x_{i,j}}(\mu)) \leq \frac{\epsilon_1}{100} \quad (9.5.44)
\]

holds for all \( a \). We call the first type a *thick* component, and the second type a *thin* component. They are not mutually exclusive, but cover all the irreducible components. By Lemma 9.5.29, thin components satisfy the requirements imposed in Proposition 9.5.28.

We first analyze the thick components. We denote the set of thick components by

\[
V_{1,i} = V_1((\Sigma_i, z_i^+), u_i) = \left\{ v \in V_T \mid \text{Diam} \left( u_i(\Sigma'_{i,v} - \bigcup_{x_{i,j}} D_{x_{i,j}}(\mu)) \right) \geq \frac{\epsilon_1}{1000} \right\}.
\]

Note that, by the monotonicity formula, if \( v \in V_1 \),

\[
\text{Area}(\Sigma'_{i,v} \setminus \bigcup_{x_{i,j}} D_{x_{i,j}}(\mu)) = \int_{\Sigma'_{i,v} \setminus \bigcup_{x_{i,j}} D_{x_{i,j}}(\mu)} u_i^* \omega \geq C \epsilon_1^2.
\]

Therefore \( \#(V_{1,i}) \) is uniformly bounded over \( i \) and *irrespective of the number of marked points* put on \( \Sigma_i^+ \). We also define two subsets of \( V_{1,i} \) by the following subsets:

\[
V_{2,i} = V_2((\Sigma_i, z_i^+), u_i) = \left\{ v \in V_{1,i} \mid \exists z_{a,j_1}, z_{a,j_2} \in \Sigma_i, v \cap z_i^+, \text{dist}(u_i(z_{a,j_1}), u_i(z_{a,j_2})) \geq \frac{\epsilon_1}{1000} \right\},
\]

\[
V_{3,i} = V_3((\Sigma_i, z_i^+), u_i) = \left\{ v \in V_{1,i} \mid \limsup_{i \to \infty} \sup_{\Sigma'_{i,v} \setminus \bigcup_{x_{i,j}} D_{x_{i,j}}(\mu)} |du_i| < \infty \right\}.
\]

(9.5.45)
Obviously we have

\[ \#(V_{2,i}) \leq \#(V_{3,i}) \leq \#(V_{1,i}). \]

**Lemma 9.5.31** Suppose there exist points

\[ p_i \in \Sigma'_i - \cup \Sigma_{x,v}(\mu) = \Sigma_i - \cup \Sigma_{x,v}(W_v(\mu)), \]

such that \( |du_i(p_i)| \to \infty \), and in particular \( v \in V_{1,i} \). Then we can take a subsequence, add marked points and take a new limit \((\Sigma_i^+, z_i^+)\) of \((\Sigma_i, z_i^+)\) such that one of the following alternatives holds:

\[
\begin{align*}
(1) & \quad \#V_1((\Sigma_i, z_i^+), u_i) > \#V_1((\Sigma_i, z_i^+), u_i) \\
(2) & \quad \#V_1((\Sigma_i, z_i^+), u_i) = \#V_1((\Sigma_i, z_i^+), u_i), \\
& \quad \#V_2((\Sigma_i, z_i^+), u_i) > \#V_2((\Sigma_i, z_i^+), u_i) \\
(3) & \quad \#V_1((\Sigma_i, z_i^+), u_i) = \#V_1((\Sigma_i, z_i^+), u_i), \\
& \quad \#V_2((\Sigma_i, z_i^+), u_i) = \#V_2((\Sigma_i, z_i^+), u_i), \\
& \quad \#V_3((\Sigma_i, z_i^+), u_i) > \#V_3((\Sigma_i, z_i^+), u_i).
\end{align*}
\]

**Proof** For the sake of notational simplicity, we denote

\[ \Sigma'_i(\mu; \text{reg}) = \Sigma'_i \cup D_{x,v}(\mu). \]

Let \( p_i \in \Sigma'_i(\mu; \text{reg}) \) be a point with

\[ |du_i(p_i)| = \sup_{\Sigma'_i(\mu; \text{reg})} |du_i| =: C_{v,i}. \]

There are two cases to consider,

1. \( C_{v,i} \text{ dist}(p_i, \partial \Sigma'_i(\text{reg})) \to \infty \)
2. \( C_{v,i} \text{ dist}(p_i, \partial \Sigma'_i(\text{reg})) \to D_v \geq 0. \)

For case (1), by the bubbling argument, there occurs a bubble around \( p_i \) by virtue of the rescaling of the domain by the order of \( C_{v,i} \), and, in particular, we can choose \( p'_i \in \Sigma'_i(\text{reg}) \) such that dist\((u_i(p_i), u_i(p'_i)) \geq \epsilon_i/2\), and dist\((p_i, p'_i) < 1/(2C_{v,i})\). Add \( p_i \) and \( p'_i \) as two new marked points in \( \Sigma'_{i,v} \setminus \cup D_{x,v}(\mu) \) and consider \((\Sigma_i, z_i^+)\) and its limit in \( \mathcal{M}_{g,k,l^+} \). Here we recall that \( \Sigma_i \) contains a subset identified with \( \Sigma'_{i,v} \setminus \cup D_{x,v}(\mu) \).

Note that, since \( p_i \) achieves the maximum of \( |du_i| \) on \( \Sigma'_{i,v}(\mu; \text{reg}) \), \((\Sigma_i, z_i^+)\) will satisfy

\[ \sup_{\Sigma'_i(\mu; \text{reg})} |\nabla u_i| < C \]

on the new bubble component of the limit by virtue of the bubbling construction. On the other hand, since the bubbling construction is localized near the
points \( p_i \) and \( p'_i \), \(|du_i|\) has the same behavior on all the components other than \( \Sigma'_{i,v} \setminus \cup D_{x,v}(\mu) \) among the components \( \Sigma'_i \). Therefore

- we obtain the inequalities \(#V_1((\Sigma_i, z^+_i), u_i) \leq #V_1((\Sigma_i, z^{++}_i), u_i)\) and

- one of the following two alternatives must hold:

\[
#V_2((\Sigma_i, z^+_i), u_i) < #V_2((\Sigma_i, z^{++}_i), u_i)
\]

or

\[
#V_3((\Sigma_i, z^+_i), u_i) < #V_3((\Sigma_i, z^{++}_i), u_i).
\]

This finishes the proof of the lemma for the first case.

Next we consider case (2), i.e., the case where we have \( p_i \in \Sigma'_{i,v}(\mu; \text{reg}) \) such that \(|du_i(p_i)| \rightarrow \infty\), with \(|du_i(p_i)| \text{dist}(p_i, W_{x,i}(\mu)) \rightarrow D_v < \infty\).

In this case, because the speed of \( p_i \) approaching the neck region is faster than the speed of bubbling, we cannot fully capture this bubble and hence cannot use the bubbling argument to select the appropriate marked points as in the first case.

However, we can still prove the following lemma using the decay estimates given in Lemma 9.5.29.

**Sublemma 9.5.32** Let \( \Sigma'_{i,v} \) be the component of \( \Sigma'_i \) containing \( p_i \). By taking a subsequence if necessary, we can choose a sequence of \( (p_{i,+}, p_{i,-}) \in \Sigma'_{i,v} \) such that

1. \(|\text{dist}(u_i(p_{i,+}), u_i(p_{i,-})) - \epsilon_1/10|\) for all \( i \) and
2. \( \lim_{i \to \infty} \text{dist}(p_{i,\pm}, p_i) = 0 \).

**Proof** Choose \( q_i \neq p_i \in \Sigma'_{i,v} \) such that \(|d(p_i, q_i)| \rightarrow 0\) and denote \( d(p_i, q_i) = \delta_i\).

Let \( 0 < \lambda < \mu \) be a sufficiently small constant fixed independently of \( i \). We claim that

\[
\limsup_{i \to \infty} \text{Diam}(u_i(D_{q_i}(\lambda))) \geq \frac{\epsilon_1}{2}.
\]

(9.5.47)

Suppose to the contrary that there exists a subsequence, again denoted by \( u_i \), such that \(|\text{Diam}(u_i(D_{q_i}(\lambda))) - \epsilon_1/2| < \epsilon_1\) for all \( i \). Choose a complex coordinate centered at \( q_i \) and regard \( D_{q_i}(\lambda) \) as a subset \( \mathbb{C} \) and \( q_i = 0 \). Consider the annulus \( \text{Ann}(\delta^2_i, 1) \) and a holomorphic embedding

\[
\psi_i : \text{Ann}(\delta^2_i, 1) \rightarrow D_{q_i}(\lambda),
\]

\[
\psi_i(z) = q_i + \frac{\lambda}{2} z \left( = \frac{\lambda}{2} z \right)
\]
Then $p_i$ lies in the image of $\psi_i$. If we write $\psi_i(z_i) = p_i$ (see Figure 9.13), where $z_i \in \text{Ann}(\delta_i^2, 1)$, then we have $|z_i|/\delta_i \sim 2/\lambda$, since we have $d(p_i, q_i) = \delta_i$ by the definition of $\delta_i$. Obviously, $\text{Diam}(u_i \circ \psi_i(\text{Ann}(\delta_i^2, 1))) \leq \epsilon_1$ by the hypothesis. Therefore we derive from Lemma 9.5.29 that we have

$$\left| \frac{\partial u_i \circ \psi_i^{-1}}{\partial z} \right| + \left| \frac{\partial u_i \circ \psi_i^{-1}}{\partial z} \right| \leq C \max \left( 1, \frac{\delta_i^2}{|z_i|^2} \right) \leq C,$$

since $\delta_i \leq \lambda$ and $0 < \lambda \ll 1$. However, we have

$$|d(u_i(p_i))| \leq \frac{|d(u_i \circ \psi_i^{-1})(\psi_i(p_i))|}{|d\psi_i(p_i)|} \leq \frac{2C}{\lambda} < \infty$$

independently of the $i$. This contradicts the hypothesis $|d(u_i(p_i))| \to \infty$. We have thus proved (9.5.47) for each fixed $\lambda > 0$.

Now choose a sequence $\lambda_m$ with $\lambda_m \to 0$. Applying (9.5.47), we can find points $q_{i,m}$ and a subsequence $u_{i,m}$ and then take a diagonal subsequence so that

$$\limsup_{m \to \infty} \text{Diam}(u_{m,m}(D_{q_{m,m}}(\lambda_m))) \geq \frac{\epsilon_1}{2}.$$

We choose any two points $p_{m}^+$ and $p_{m}^-$ on $D_{q_{m,m}}(\lambda_m)$ such that

$$\text{dist}(u_{m,m}(p_{m}^+), u_{m,m}(p_{m}^-)) \geq \frac{\epsilon_1}{4}.$$

Recall that both of the points $p_{m}^+$ and $p_{m}^-$ are close to $q_{m,m}$ and $d(p_{m}^+, q_{m,m}) \to 0$. This finishes the proof of Sublemma 9.5.32.

Now we go back to the proof of Lemma 9.5.31. Take $p_{i,+}$ and $p_{i,-}$ in Sublemma 9.5.32 as two new marked points. Denote by $(\Sigma_\infty^{++}, z_\infty^{++})$ the limit of the resulting curves $(\Sigma_i, z_i^{++})$ in $\overline{M}_{g,|\Gamma_0|}$. Since $p_{i,\pm}$ are uniformly far away from other marked points, $(\Sigma_\infty^{++}, z_\infty^{++})$ is obtained by adding one $CP^1$ to $\Sigma_i$ with two new marked points on $CP^1 = \Sigma_{\text{new},\infty}$. By construction we have

$$\Sigma_\infty^{++} = \Sigma_\infty \cup \Sigma_{\text{new},\infty},$$

where $\Sigma_{\text{new},\infty}$ is attached to $\Sigma_i$. In this process, nothing has changed as in the first case for the components other than $\nu$ and $\nu_{\text{new}}$ from $\Sigma_\infty^{++}$. This implies that,
if \( w \in V_j((\Sigma_i, z_i^+), u_i), w \neq v \), then \( w \in V_j((\Sigma_i, z_i^{++}), u_i) \). If \( w \in V_j((\Sigma_i, z_i^{++}), u_i) \) and \( w \) is equal to neither \( v \) nor \( v_{\text{new}} \), we have \( w \in V_j((\Sigma_i, z_i^+), u_i) \).

Since \( \text{dist}(u_i(p_{a,-}), u_i(p_{a,+})) > \epsilon_1/20 \), \( v_{\text{new}} \) is a non-constant component and in particular \( u_i(\Sigma_{a,\text{new}} \setminus D_{v_{\text{new}}} (\mu)) \geq \epsilon_1/1000 \). Hence we have \( v_{\text{new}} \in V_1((\Sigma_i, z_i^{++}), u_i) \).

Furthermore, \( v_{\text{new}} \) has only two marked points, \( p_{a,+} \) and \( p_{a,-} \), and one singular point, we may assume that \( \text{dist}(p_{a,+}, p_{a,-}) > \epsilon_1/2 \) with respect to the standard metric on \( \mathbb{C}P^1 \). Hence we do indeed have that \( v_{\text{new}} \in V_2((\Sigma_i, z_i^{++}), u_i) \).

There are three cases to examine for the component \( v \).

1. For \( v \notin V_1((\Sigma_i, z_i^+), u_i) \), we have
   \[ \#V_1((\Sigma_i, z_i^+), u_i) < \#V_1((\Sigma_i, z_i^{++}), u_i) \]
   since \( v_{\text{new}} \in V_1((\Sigma_i, z_i^{++}), u_i) \).

2. For \( v \in V_1 \setminus V_2((\Sigma_i, z_i^+), u_i) \), we have
   \[ \#V_1((\Sigma_i, z_i^+), u_i) \leq \#V_1((\Sigma_i, z_i^{++}), u_i) \]
   \[ \#V_2((\Sigma_i, z_i^+), u_i) < \#V_2((\Sigma_i, z_i^{++}), u_i) \]
   because we have \( v_{\text{new}} \in V_2 \).

3. For \( v \in V_2((\Sigma_i, z_i^+), u_i) \), it is also the case that \( v \in V_2((\Sigma_i, z_i^{++}), u_i) \). Since we have \( v_{\text{new}} \in V_2 \), this implies that
   \[ V_2((\Sigma_i, z_i^+), u_i) < V_2((\Sigma_i, z_i^{++}), u_i) \].

Therefore, in all cases, one of \( V_k \) for \( k = 1, 2, 3 \) must strictly increase. \( \square \)

Now we are ready to wrap up the proof of Proposition 9.5.28.

As we mentioned before, we have the universal bound for \( \#(V_{1,i}) \) depending only on \( \beta \) but independent of \( i \) and of the number of marked points. On the other hand, Lemma 9.5.31 implies that, whenever there is a component on which the derivative of the map blows up, we can add marked points and take a new sequence for which one of the numbers \( \#(V_{k,i}) \) grows for \( k = 1, 2, 3 \) without changing the numbers \( \#(V_{l,i}) \) for \( 1 \leq l \leq k - 1 \). By the universal bound for \( \#(V_{1,i}) \), this process must stop, i.e., eventually the derivatives will be bounded on all the components away from the prescribed neck regions, if we increase the size of \( \#A \) in the choice of added marked points \( w_{i} = \{ w_{i,a} \}_{a \in A} \). This finishes the proof of Proposition 9.5.28. \( \square \)

**Convergence on the neck regions**

Now we examine the behavior of the diameter of the image of \( u_i \) on each neck region of \( \Sigma_i \). Our goal will be to show that, after adding additional marked
points, if necessary, on the neck regions, thereby increasing the size of the indexing set $A$ further and extracting relevant bubbles away, the diameter of each neck region will converge to 0. By letting $\mu \to 0$ and taking a diagonal subsequence, we may assume by Proposition 9.5.28 that $u_i : (\Sigma_i, z_i^+) \to M^+$ converges on compact subsets of $\Sigma_i \setminus \text{Sing}(\Sigma_i)$.

Now we consider the behavior of the sequence on the neck region as $\mu \to 0$. Fix a sufficiently small $\lambda_0 > 0$ and consider the annuli

$$W_{e,i}^+(\lambda_0) = D_{x_{i,j}}(\lambda_0) \# (\alpha_{e,i}) D_{x_{i,j}}(\lambda_0).$$

Then we choose a sequence $\lambda_i \to 0$, e.g.,

$$\lambda_i = |\alpha_{e,i}|^{1/8} \quad (= R_{e,i}^{-1/4})$$

and consider the sequence

$$W_{e,i} = D_{x_{i,j}}(\lambda_i) \# (\alpha_{e,i}) D_{x_{i,j}}(\lambda_i).$$

We have conformal isomorphisms

$$W_{e,i}^+ = [-L_{e,i}^+, L_{e,i}^+] \times S^1,$$

$$W_{e,i} = [-L_{e,i}, L_{e,i}] \times S^1,$$

where

$$L_{e,i}^+ \sim \ln \left( \frac{\lambda_0}{|\alpha_{e,i}|^{1/4}} \right) = \ln \lambda_0 - \frac{1}{4} \ln |\alpha_{e,i}|,$$

$$L_{e,i} \sim \ln \left( \frac{\lambda_i}{|\alpha_{e,i}|^{1/4}} \right) = - \frac{1}{8} \ln |\alpha_{e,i}|.$$

In particular, $L_{e,i}^+ \to \infty$ and $L_{e,i}^+ - L_{e,i} \to \infty$. See Figure 9.14.

It follows from Proposition 9.5.28 that, on each compact subset $K \subset \Sigma^+ \setminus \text{Sing}(\Sigma^+)$, $u_i$ converges in $C^\infty$ topology. For we can choose a subsequence $u_i$ as $i \to \infty$ for given $\mu > 0$ and then let $\mu \to 0$ and take a diagonal subsequence and prove that $u_i$ converges on $K$. Here the last statement makes sense because for any compact subset $K \subset \Sigma^+ \setminus \text{Sing}(\Sigma^+)$, we can choose compact subsets $K_i \subset \Sigma^+ \setminus \text{Sing}(\Sigma^+)$ such that $K_i \to K$ and $(u_i, K_i) \to (u_\infty, K)$ in the obvious sense.

In particular, the maps $u_i|_{\partial W_{e,i}^+}^+(\lambda_0)$ converge as $i \to \infty$ since $u_i$ converges on the compact subsets

$$K_i = \Sigma_i \setminus \bigcup_{e} W_{e,i}^+(\lambda_0),$$

$$K = \Sigma^+ \setminus \bigcup_{e} W_{e,\infty}^+(\lambda_0) \subset \Sigma^+ \setminus \text{Sing} \Sigma^+.$$
Therefore we can apply the same argument as was used for the region outside $W(\mu)$ above, by adding more marked points on the neck $W_{e,i}^+ (\lambda_0)$ and splitting bubbles off, and achieve the $C^1$ bound
\[
\left| \frac{\partial u_i}{\partial \tau} \right| + \left| \frac{\partial u_i}{\partial t} \right| < C < \infty
\] (9.5.48)
on $W_{e,i}^+ (\lambda_0)$ uniformly over all $i$ for some constant $C$ that is independent of $i$ after choosing a subsequence.

**Lemma 9.5.33** We can make a suitable choice of additional marked points, take a new limit for a subsequence and split off bubble components so that the following statements hold. For each given $\epsilon > 0$, we can choose a positive real number $L(\epsilon)$ and an integer $i(\epsilon)$ so that

(i) $L(\epsilon), i(\epsilon) \to \infty$ as $\epsilon \to 0$ and

(ii) if $\tau \in [-L_{e,i} + L(\epsilon), L_{e,i} - L(\epsilon)]$ and $i > i(\epsilon)$, then
\[
\left| \frac{\partial u_i}{\partial \tau} (\tau, t) \right| + \left| \frac{\partial u_i}{\partial t} (\tau, t) \right| \leq \epsilon
\]
for all $i$.

**Proof** We prove this by contradiction. Suppose to the contrary. Then there exists $\epsilon_2 > 0$ such that, for any choice of $L_2 \in \mathbb{R}, i_2 \in \mathbb{N}$, there exists $i > i_2$ such that
for some $T_i \in [-L_{e,i} + L_2, L_{e,i} - L_2]$, $t_i \in S^1$.

Denote also by $T_i$ the translation of cylinder by $T_i$, and then consider $u_i \circ T_i|_{[-L_2, L_2] \times S^1}$ as $L_2 \to \infty$. Since $T_i \in [-L_{e,i} + L_2, L_{e,i} - L_2]$ and $L_{e,i} - L_2 \to \infty$ as $i \to \infty$, we have $L_{e,i} - L_2 > L_2$ eventually. (See Figure 9.15.) Therefore the domain of the map $u_i \circ T_i$ will eventually contain $[-L_2, L_2] \times S^1$. Then by (9.5.48) $u_i \circ T_i$ has a uniform derivative thereon and $|\frac{\partial u_i}{\partial \tau}(T_i, t_i)| + |\frac{\partial u_i}{\partial t}(T_i, t_i)| \geq \epsilon_2$. Therefore, we can extract a non-constant bubble around $T_i$ for $u_i$ by letting $L_2 \to \infty$. Then we may add two additional marked points suitably on the cylinder, and take convergence of the new stable curve on the Deligne–Mumford space. This process will increase the number of components contained in $V_1$. Because of the uniform bound on $\#V_1$ which is independent of the number $\#A$ of added marked points, after repeating this process enough times, we will obtain the lemma. □

The following lemma is easy to prove, so the proof is left as an exercise.

**Lemma 9.5.34** Under the assumption of Lemma 9.5.33, there exist constants $C'$, $k' > 0$ independent of $i$ such that

$$\operatorname{Diam}(u_i([-L_{e,i} + B, L_{e,i} - B] \times S^1)) \leq C' e^{-k'B}$$

for all $0 < B \leq L_{e,i} - 1$.

**Exercise 9.5.35** Prove this lemma. (Hint. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies $\frac{d^2 f}{dt^2} \geq \lambda f$ for some $\lambda > 0$. Let $L > 0$ be given. Then, if $|f(\pm L)| \leq C_0$, $f$ satisfies the inequality

$$|f(t)| \leq C_1 e^{-\sqrt{\lambda} \operatorname{dist}(t, \pm L)}$$

(9.5.49)

for $t \in [-L, L]$. Or see the proof of Proposition 14.1.5.)
We take $B = \frac{1}{2} L_{e,i}$, which is certainly smaller than $L_{e,i} - 1$ for all sufficiently large $i$, and so obtain
\[
\text{diam}(u_i([-L_{e,i}/2, L_{e,i}/2] \times S^1)) \leq C' e^{-(k'/2)L_{e,i}}
\]
for all $i$ from this lemma. Therefore,
\[
\lim_{\mu \to 0} \limsup_{i \to \infty} \text{Diam}(u_i(W_{e,i}(\mu))) = 0
\]
for any nodal point $x = x_e$. On combining this with the convergence on compact sets of $u_i$ away from Sing($\Sigma_\infty$) (Proposition 9.5.28), we have proved
\[
\text{s-lim}_{i \to \infty} [(\Sigma_i, z_i^+), u_i] = [(\Sigma_\infty, z_\infty^+), u_\infty], \quad z_i^+ = z_\infty \cup w_i,
\]
for a suitable choice of additional marked points $w_i = \{w_{i,a}\}_{a \in A}$.

**Forgetting the added marked points**

Now we would like to go back to the convergence of the original sequence $((\Sigma_i, z_i), u_i)$ by removing the added marked points $w_i$. We need to be able to define
\[
\text{forget}_{m+n,M}[(\Sigma_\infty, z_\infty^+), u_\infty] \in \overline{\mathcal{M}}_{g,m}(M, J; \beta),
\]
where $n = \#A$.

**Definition 9.5.36** We call a constant component $\Sigma_\infty, v$ of $\Sigma_\infty$ a **dead component** if it becomes unstable after removal of all the added auxiliary marked points, and call it a **ghost component** if it remains stable after removal of all the added auxiliary marked points. See Figure 9.16.

We remark that any dead component can have at most two singular points. Consider the union of all dead components. Denote by $Y = \bigcup_{l \in L} \Sigma_{\infty, v_l}$ one of its connected components. Then $Y$ is mapped to a point by $u_\infty$. Then we consider the corresponding union associated with $u_i$,
\[
\bigcup_{l \in L} \Sigma_{i, v_l} \setminus \bigcup_x D_{\lambda_i}(\lambda_i),
\]
and add back the necks to each singular point of $Y$ (also the neck at the intersection of $Y$ and nearby stable components). We denote the resulting open surface by $Y_i$.

Then the topological type of $Y_i$ must be either a disc or a cylinder. If $Y_i$ is of disc type, it can carry at most one of the original marked points. If $Y_i$ is of annulus type, $Y_i$ cannot contain any of them.
We remove all the added marked points from $Y_i$. We know that $\text{Diam}(u_i(Y_i)) \to 0$ as $i \to \infty$. It follows from these considerations that the resulting pre-stable map is stable.

Denote the resulting stable map by $((\Sigma_i, z_i^+), u_i)$. We then remove all the added marked points from $Y \subset \Sigma_{\infty}$, contract all the components in $Y$ and denote the resulting stable map by $((\Sigma_{\infty}, z_{\infty}^+), u_{\infty})$.

Then we can remove all the added marked points safely from the non-constant component of $((\Sigma_{\infty}, z_{\infty}^+), u_{\infty})$ and correspondingly from $((\Sigma_i, z_i^+), u_i)$. We denote by $((\Sigma_{\infty}, z_{\infty}^{++}), u_{\infty})$ and $((\Sigma_i, z_i^{++}), u_i)$, respectively, the resulting stable maps. By construction, we have $[(\Sigma_i, z_i^{++}), u_i] \in \mathcal{M}_{g,m}(M, \omega; A)$ and $[(\Sigma_{\infty}, z_{\infty}^{++}), u_{\infty}] = [(\Sigma_i, z_i), u_i]$ obviously by construction. Hence, we have

\[
\text{forget}_{m+n,m}[(\Sigma_i, z_i^+), u_i] = [(\Sigma_i, z_i), u_i],
\]

and

\[
\text{forget}_{m+n,m}[(\Sigma_{\infty}, z_{\infty}^{++}), u_{\infty}] = [(\Sigma_{\infty}, z_{\infty}^{++}), u_{\infty}]
\]
is well defined in $\mathcal{M}_{g,m}(M, J; \beta)$. Then, by the definition of the convergence given in Definition 9.5.20, we have

\[
\lim_{i \to \infty}[(\Sigma_i, z_i), u_i] = [(\Sigma_{\infty}, z_{\infty}^{++}), u_{\infty}].
\]

This finally finishes the proof of compactness.

### 9.5.4 The Hausdorff property

In this subsection, we examine the Hausdorff property of $\mathcal{M}_{g,m}(M, J; \beta)$. For this purpose, we recall the definition of the forgetful maps

\[
\text{forget}_{m+n,m} : \mathcal{M}_{g,m+n}(M, J; \beta) \to \mathcal{M}_{g,m}(M, J; \beta).
\]

Let $((\Sigma, z), u)$ be an element of $\mathcal{M}_{g,m+n}(M, J; \beta)$ and $u = (u_v)_{v \in V_f}$ be its decomposition. After we have forgotten the last $n$ marked points, if an irreducible component $(\Sigma_v, u_v)$ becomes unstable (i.e., if the genus of $\Sigma_v$ is zero and the

---

**Figure 9.16 Dead components.**

[Diagram showing stable and dead components with annotation D near the end of the stable components and Y_i representing a marked point or component.]
map \( u_v \) is constant), then we just contract \( \Sigma_v \) to a point and remove the component \((\Sigma_v, u_v)\). If \((\Sigma_v, u_v)\) remains stable after the forgetting of the marked points, we leave \((\Sigma_v, u_v)\) as it is, just removing the designated added marked points from \( \Sigma_v \).

**Theorem 9.5.37**  The stable map topology on \( \overline{M}_{g,m}(M, J; \beta) \) is Hausdorff.

**Proof**  To prove the Hausdorff property of \( \overline{M}_{g,m}(M, J; \beta) \), we have to prove that the diagonal of \( \overline{M}_{g,m}(M, J; \beta) \times \overline{M}_{g,m}(M, J; \beta) \) is a closed subset. We recall that the set of stable maps \( \overline{M}_{g,m}(M, J; \beta) \) is metrizable. Therefore it will suffice to prove the following statement.

Let \((\Sigma, z_i, u_i), (\Sigma, z'_i, u'_i) \in \overline{M}_{g,m}(M, J; \beta)\) be two representatives of a convergent sequence of stable maps in \( \overline{M}_{g,m}(M, J; \beta) \), i.e., \([\Sigma, z_i, u_i] = [\Sigma, z'_i, u'_i]\).

By the definition of convergence in the stable map topology, there exist stabilizations of domains \((\Sigma_i, z^+_i)\) and \((\Sigma'_i, z'^+_i)\), such that

\[
\text{s-lim}_{i \to \infty} [\Sigma, z^+_i, u_i] = [\Sigma, z^+_{\infty}, u_{\infty}] \quad \text{in} \quad \overline{M}^s_{g,m+n}(M, J; \beta),
\]

\[
\text{s-lim}_{i \to \infty} [\Sigma, z'^+_i, u'_i] = [\Sigma, z'^+_{\infty}, u'_{\infty}] \quad \text{in} \quad \overline{M}^s_{g,m+n'}(M, J; \beta)
\]

with

\[
[\Sigma, z, u] = \text{forget}_{m+n;m} [\Sigma, z^+_{\infty}, u_{\infty}],
\]

\[
[\Sigma, z', u'] = \text{forget}_{m+n';m} [\Sigma, z'^+_{\infty}, u'_{\infty}].
\]

By the convergence hypothesis in the sense of Definition 9.5.20, these are well defined. We need to prove that the two stable maps \((\Sigma, z, u)\) and \((\Sigma, z', u')\) are isomorphic.

By the definition of the equivalence classes, we have a biholomorphism \( \phi_i : (\Sigma_i, z_i) \rightarrow (\Sigma_i, z'_i) \) and \( u'_i = u_i \circ \phi_i^{-1} \) for each \( i \in \mathbb{N} \). We consider the subset \( z''_i = z^+_i \cup z'^+_i \subset \Sigma_i \) consisting of added marked points, i.e.,

\[
z''_i = \{ z_i, z^+_i \setminus z_i, z'^+_i \setminus z_i \}. \]

By perturbing the points in \( z''_i \setminus z'_i \), if necessary, we may assume that the points in \( z''_i \) are all different.

Then obviously the domains of the stable maps \((\Sigma_i, z''_i, u_i)\) are stable and hence they define elements in \( \overline{M}^s_{g,m+n+n'}(M, J; \beta) \subset \overline{M}_{g,m+n+n'}(M, J; \beta) \). By the compactness of \( \overline{M}_{g,m+n+n'}(M, J; \beta) \) and by the definition of its topology, we can add \( n'' \) additional marked points and achieve
9.5 Stable map and stable map topology

\[ s\text{-lim}_{i \to \infty} \left[ (\Sigma_i, z_i'''), u_i \right] = \left[ (\Sigma_{\infty}', z_{\infty}'''), u_{\infty}' \right] \quad (9.5.50) \]

for some \( [\Sigma_{\infty}, z_{\infty}'''), u_{\infty}] \in \overline{M}_{g,m+n+n'+n''}^M (M, J; \beta) \) after choosing a subsequence if necessary. By construction, we have

\[ \text{forget}^i_{m+n+n'+n''; m+n} \left[ (\Sigma_i, z_i'''), u_i \right] = \left[ (\Sigma_i, z_i^+), u_i \right]. \]

The convergence (9.5.50) in particular guarantees that the marked points are all uniformly separated for sufficiently large \( i \). We also recall that

\[ s\text{-lim}_{i \to \infty} \left[ (\Sigma_i, z_i^+), u_i \right] = \left[ (\Sigma_{\infty}, z_{\infty}^+), u_{\infty} \right]. \]

Similarly, we have

\[ \text{forget}^i_{m+n+n'+n''; m+n} \left[ (\Sigma_i, z_i'''), u_i \right] = \left[ (\Sigma_i', z_i^+), u_i' \right] \]

and

\[ s\text{-lim}_{i \to \infty} \left[ (\Sigma_i', z_i^+), u_i' \right] = \left[ (\Sigma_{\infty}', z_{\infty}^+), u_{\infty}' \right]. \]

From these, we derive

\[ \text{forget}^i_{m+n+n'+n''; m+n} \left[ (\Sigma_{\infty}', z_{\infty}'''), u_{\infty}' \right] = \left[ (\Sigma_{\infty}, z_{\infty}^+), u_{\infty} \right], \]

\[ \text{forget}^i_{m+n+n'+n''; m+n} \left[ (\Sigma_{\infty}', z_{\infty}'''), u_{\infty}' \right] = \left[ (\Sigma_{\infty}', z_{\infty}^+), u_{\infty}' \right]. \]

On the other hand, we had

\[ \text{forget}^i_{m+n; m} \left[ (\Sigma_{\infty}, z_{\infty}^+), u_{\infty} \right] = \left[ (\Sigma, z), u \right], \]

\[ \text{forget}^i_{m+n'; m'} \left[ (\Sigma_{\infty}', z_{\infty}'), u_{\infty}' \right] = \left[ (\Sigma', z'), u' \right]. \]

Therefore we have obtained

\[ [(\Sigma, z), u] = \text{forget}^i_{m+n; m} \circ \text{forget}^i_{m+n+n'+n''; m+n} \left[ (\Sigma_{\infty}', z_{\infty}'), u_{\infty}' \right], \]

\[ [(\Sigma', z'), u'] = \text{forget}^i_{m+n'; m} \circ \text{forget}^i_{m+n+n'+n'; m+n} \left[ (\Sigma_{\infty}', z_{\infty}'), u_{\infty}' \right]. \]

We also have the following general lemma, which is the analog of Lemma 9.3.22 for the stable map moduli spaces.

**Lemma 9.5.38** For any \( a, b, c \in \mathbb{N} \),

\[ \text{forget}_{a+b; a} \circ \text{forget}_{a+b+c; a+b} = \text{forget}_{a+b+c; a} \]

on \( \overline{M}_{g,a+b+c}(M, J; \beta) \).
By applying this lemma with \( a = m, b = n, c = n' + n'' \) or \( a = m, b = n', c = n' + n'' \) to the right-hand side of the equation immediately above the lemma, we obtain

\[
[(\Sigma, z), u] = [(.\Sigma', z'), u']
\]

because both are obtained by forgetting marked points of the same enumerations from the same stable map \([(.\Sigma''_{\infty}, z''_{\infty}), u''_{\infty}] \in \overline{M}_{g,m+n+n''}(M, J; \beta)\). This proves that the diagonal of \( \overline{M}_{g,m}(M, J; \beta) \times \overline{M}_{g,m}(M, J; \beta) \) is a closed subset. This finishes the proof of the Hausdorff property. \( \square \)
Fredholm theory

In this chapter, we explain the appropriate functional analytic setting used for the crucial transversality study of the moduli space of pseudoholomorphic maps. In physics language, this corresponds to the off-shell description of the moduli problem. For this purpose, it is important to observe that the moduli space of $J$-holomorphic maps can be regarded as the zero set of a Fredholm section of an infinite-dimensional Banach vector bundle over a Banach manifold.

10.1 A quick review of Banach manifolds

We first briefly summarize the theory of Banach manifolds and the Sard–Smale theorem (Sm65).

Let $\mathbb{B}_1$, $\mathbb{B}_2$ be two Banach spaces. We denote by $L(\mathbb{B}_1, \mathbb{B}_2)$ the set of bounded linear maps equipped with norm topology.

**Definition 10.1.1** A bounded linear map $L : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ is called a Fredholm operator if it satisfies the following three conditions:

1. it has a closed range
2. $\dim(\ker L) < \infty$
3. $\dim(\text{coker} L) < \infty$.

The Fredholm index, denoted by $\text{Index } L$, is defined to be

$$\text{Index } L = \dim(\ker L) - \dim(\text{coker} L).$$

The following is a standard theorem that can be proved from the definition and the basic theorems in functional analysis of Banach spaces. (See, e.g., (Rud73), (La83) for the details of the proof.)
Proposition 10.1.2  The index function $L \mapsto \mathbb{Z}$ is a continuous function with respect to the norm topology on $L(\mathbb{B}_1, \mathbb{B}_2)$ (and hence invariant under the continuous homotopy of Fredholm operators). Furthermore, the set of Fredholm operators forms an open subset in $L(\mathbb{B}_1, \mathbb{B}_2)$.

Proof  We outline the main steps of the proof, leaving the details as an exercise. First the finite dimensionality of $\ker L$ and $\coker L$ provides a splitting

$$\mathbb{B}_1 = \mathbb{B}_1' \oplus \ker L, \quad \mathbb{B}_2 = \text{image } L \oplus C_2,$$

with respect to which $L$ can be written as

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix},$$

where $L_{11} : \mathbb{B}_1' \to \text{image } L$ is an isomorphism and $L_{12}, L_{21}, L_{22}$ are compact operators.

Exercise 10.1.3  Complete the proof, starting from the above preparation. □

Now let $f : U_1 \to U_2$ be a continuous map, where $U_i \subset \mathbb{B}_i$ are open subsets of the respective $\mathbb{B}_i$. The map $f$ is called $C^k$ if it is so in the sense of calculus. More precisely, we have the following.

Definition 10.1.4  A continuous map $f : U_1 \to U_2$ is called differentiable at $x \in U_1$, if there exists a bounded linear map $A : \mathbb{B}_1 \to \mathbb{B}_2$ such that

$$f(x + h) - f(x) = A \cdot h + c(h)$$

for all $h \in \mathbb{B}_1$, where $c$ is a continuous map in a neighborhood of $0 \in \mathbb{B}_1$ such that $\lim_{h \to 0} \|c(h)\|_{\mathbb{B}_2}/\|h\|_{\mathbb{B}_1} = 0$. We call such a linear operator $A$ the derivative of $f$ at $x$ and denote this by $df(x)$. We say that $f$ is continuously differentiable, or of class $C^1$, if the map

$$df : \mathbb{B}_1 \to L(\mathbb{B}_1, \mathbb{B}_2)$$

is continuous with respect to the norm topology of $L(\mathbb{B}_1, \mathbb{B}_2)$. Proceeding inductively, we define the $r$th derivative

$$d^r f = d(d^{r-1} f) : U_1 \to L'(\mathbb{B}_1, \mathbb{B}_2) = L'(\mathbb{B}^r_1, \mathbb{B}_2)$$

and a map of class $C^r$. We call a map smooth (or infinitely differentiable) if it is of class $C^r$ for all $r = 0, \ldots, \to \infty$.

We refer readers to (La02) and (AMR88) for more details on Banach differentiable manifolds and their detailed calculus.
Definition 10.1.5 A Banach manifold is a second-countable topological space that carries an atlas \(\{(U_\alpha, \phi_\alpha)\}\) consisting of local charts \(\phi_\alpha : U_\alpha \to \mathbb{B}\), where \(\mathbb{B}\) is a Banach space such that

\[
\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)
\]
is smooth (or \(C^k\)).

One can also define the notion of Banach vector bundles \(E \to M\) over a Banach manifold that is locally trivial and each fiber \(E_x\) is a Banach space. As in the finite-dimensional case, we can define the notion of tangent bundles \(TM \to M\) as a vector bundle. We can also consider the notion of connections and their associated covariant derivatives.

Definition 10.1.6 Let \(k \geq 1\). A \(C^k\)-map \(f : M \to N\) is called a Fredholm map if \(T_x f : T_x M \to T_{f(x)} N\) is a Fredholm operator for all \(x \in M\).

By definition and by the homotopy invariance of the index, the function

\[
x \mapsto \text{Index } T_x f
\]
is locally constant, and hence constant on each path-component of \(M\). The following basic theorem and corollaries were proved by Smale (Sm65).

Theorem 10.1.7 (Sard–Smale theorem) Let \(M\) be connected and denote by index \(f\) the common integer \(\text{Index } T_x f\) for \(x \in M\). Let \(f : M \to N\) be a \(C^k\) Fredholm map with \(k > \max\{\text{index } f, 0\}\). Then the set of regular values of \(f\) is a residual subset of \(N\).

We state some corollaries of this theorem.

Corollary 10.1.8 If \(f : M \to N\) is a Fredholm map of negative index, then its image contains no interior point.

Corollary 10.1.9 Let \(M\) be connected and let \(f : M \to N\) be a Fredholm map. Then, for almost all \(y \in N\), the pre-image \(f^{-1}(y)\) is either empty or a smooth submanifold of dimension given by \(\text{index } f = \dim \ker T_y f\).

Definition 10.1.10 Let \(E\) be a Banach vector bundle over \(M\) modeled by a Banach space \(\mathbb{B}\). Let \(s : M \to E\) be a \(C^1\) section. Then \(s\) is said to be a Fredholm section if the covariant derivative \(\nabla s(x) : T_x M \to E_x\) is Fredholm with respect to a (and hence any) connection \(\nabla\).
Note that the linear operator $\nabla s(x)$ is independent of the choice of the connection $\nabla$ at a zero point $x$ with $s(x) = 0$. We call this common operator $\nabla s(x)$ the (covariant) linearization of $s$ at a zero $x$, and denote $D_s(x) : T_xM \to E_x$.

**Proposition 10.1.11** Let $s$ be a Fredholm section and $s \pitchfork o_E$, where $o_E$ is the zero section of $E$. Then the zero set $s^{-1}(o_E) \cong \text{Im } s \cap o_E$ is a smooth submanifold of $M \equiv o_E$. The dimension of $s^{-1}(o_E)$ stays constant on each connected component of $M$.

Next we recall some basic facts regarding the elliptic differential operators acting on the space of smooth sections. Let $E_1$, $E_2 \to M$ be two vector bundles over a finite-dimensional manifold $M$.

1. A linear operator $L : C^\infty(E_1) \to C^\infty(E_2)$ is called local if $\text{supp}(Ls) \subset \text{supp}(s)$ for any $s \in C^\infty(E_1)$. Any local linear operator is a differential operator (Peetre’s theorem (Pe59)).

2. An equivalent, but purely algebraic, description of linear differential operators is as follows: an $\mathbb{R}$-linear map $L$ is a $k$th-order linear differential operator, if for any $k + 1$ smooth functions $f_0, \ldots, f_k \in C^\infty(M)$ we have

$$[f_k, [f_{k-1}, \cdots [[f_0, P] \cdots]]] = 0. \tag{10.1.1}$$

Here the bracket $[f, L] : \Gamma(E) \to \Gamma(F)$ is defined as the commutator $[f, L](s) = L(f \cdot s) - f \cdot L(s)$.

3. The order($L$) is the minimal possible $k$ for which (10.1.1) holds for all choices of $f_0, \ldots, f_k \in C^\infty(M)$.

4. Any local operator of order $k$ can be expressed as $\sum_{|I|\leq k} A_I \partial/\partial x^I$, where $A_I \in \text{Hom}(E_1, E_2)$ and the top order part is independent of trivialization in the following sense: replacing $\partial/\partial x^i$ by $\xi^i \in T^*M$ in

$$\frac{\partial}{\partial x^l} := \left(\frac{\partial}{\partial x^1}\right)^{k_1} \left(\frac{\partial}{\partial x^2}\right)^{k_2} \cdots \left(\frac{\partial}{\partial x^n}\right)^{k_n} \text{ with } |I| = \sum_{i=1}^n k_i = k,$$

we define the (principal) symbol map of $L$ by

$$\sigma(L)(\xi) := \sum_{|I|=\text{order } L} A_I \xi^I. \tag{10.1.2}$$

Here $\pi : T^*M \to M$ is the canonical projection and $\pi^*E_i$ are the pull-back bundles on $T^*M$. Then this map defines a smooth section of

$$\text{Hom}(\pi^*E_1, \pi^*E_2) \to T^*M$$

such that $\sigma(L)(\xi) : E_{1,\pi(\xi)} \to E_{2,\pi(\xi)}$ is a linear map.
A differential operator $L : C^\infty(E_1) \to C^\infty(E_2)$ is called elliptic if $\sigma(L)(\xi)$ is an isomorphism for all $\xi \in T^* M \setminus \{0\}$.

Note that $L$ can be elliptic only when $\text{rank}(E_1) = \text{rank}(E_2)$, since the linear map $\sigma(L)(\xi)$ can be an isomorphism only for such cases.

The following is the basic a-priori estimate that applies to any elliptic differential operator $L$ (defined on a compact manifold without boundary).

**Proposition 10.1.13** Let $M$ be a compact manifold without a boundary. Suppose that the operator $L : C^\infty(E_1) \to C^\infty(E_2)$ is elliptic. Then, for any $\xi \in C^\infty(E_1)$,

$$\|\xi\|_{s+k,p} \leq C \left( \|L\xi\|_{s,p} + \|\xi\|_{s+k-1,p} \right)$$

for all $s$ and $1 < p < \infty$, where $k = \text{order}(L)$. In particular, $L$ extends to a bounded linear map $W^{k+s,p}(E_1)$ to $W^{s,p}(E_2)$, which is Fredholm.

**Remark 10.1.14** When $M$ has a boundary, one could define an elliptic boundary-value problem as the pair of differential operators

$$(L, B) : C^\infty(E_1) \to C^\infty(E_2) \times C^\infty(i^* E_2).$$

where $B : C^\infty(E_1) \to C^\infty(F_2)$ is a trace operator and $F_2 \to \partial M$ is a subbundle of $E_2|_{\partial M}$ on $\partial M$ with order $B \leq \text{order } L - 1$ such that the a-priori estimate

$$\|\xi\|_{s+k,p} \leq C \left( \|L\xi\|_{s,p} + \|\xi\|_{s+k-1,p} + \|B(\xi \circ i)\|_{s+l-1-1/p,p} \right)$$

holds for all $s$ and $1 < p < \infty$. See (ADN59) and (ADN64) for the derivation of a-priori estimates for the general elliptic boundary-value problem, i.e., for the operator $(L, B)$ satisfying Lopatinski–Shapiro conditions (see, e.g., (ADN64) for the explanation).

In this regard, the Riemann–Hilbert problem

$$\bar{\partial} \xi = 0, \quad \xi(\theta) \in \tau(\theta),$$

where $\tau \to \partial \Sigma$ is a totally real subbundle of $\mathbb{C}^n$, is an elliptic boundary-value problem. This is because the problem can be turned into a boundary-value problem $B\xi|_{\partial \Sigma} = 0$ for some zeroth-order operator $B$.

**Exercise 10.1.15** Give a precise definition of the zeroth-order operator $B$ above in the expression $B\xi|_{\partial \Sigma} = 0$. 
**10.2 Off-shell description of the moduli space**

We want to consider a $J$-holomorphic map $u: (\Sigma, j) \to (M, J)$ as a zero of a Fredholm section of some Banach vector bundle over $\mathcal{F} := \mathcal{C}^\infty(\Sigma, M)$.

First we assume that $\Sigma$ has no boundary. We will later indicate the changes needed in order to handle the case with boundary.

$\mathcal{C}^\infty(\Sigma, M)$ is not actually a Banach manifold with respect to $\mathcal{C}^\infty$ topology. It is only a Fréchet manifold. To make a Banach manifold out of $\mathcal{C}^\infty(\Sigma, M)$, we need to enlarge it to the space $F_k := W^{k,p}(\Sigma, M)$ of Sobolev maps for $k - 2/p > 0$ (respectively the Hölder space $C^{k,\alpha}(\Sigma, M)$), by taking the completion of $\mathcal{C}^\infty(\Sigma, M)$ with respect to the $W^{k,p}$-norm (respectively the $C^{k,\alpha}$ Hölder norm). We will work only with the Sobolev spaces and begin our discussion by recalling the notion of $W^{k,p}$-maps into a smooth manifold $M$. The easiest and quickest way to write down its definition is to use Nash’s isometric embedding theorem (Nas56).

Fix an isometric embedding $(M, g) \hookrightarrow \mathbb{R}^N$ for a sufficiently large integer $N$ and regard $M$ as a closed submanifold $\mathbb{R}^N$. Then we may regard $u: \Sigma \to M \hookrightarrow \mathbb{R}^N$ as an $\mathbb{R}^N$-valued function.

**Definition 10.2.1**  Let $k - 2/p \geq 0$. We define

$$W^{k,p}(\Sigma, M) := \{u \in W^{k,p}(\Sigma, \mathbb{R}^N) \mid u(z) \in M \text{ a.e.}\}$$

and call any element $u$ in $W^{k,p}(\Sigma, M)$ a $W^{k,p}$-map to $M$.

It is easy to check that as long as we choose $k - 2/p > 0$, for which the map $u$ is continuous, this definition is independent of the embedding and hence is well defined on a manifold.

**Remark 10.2.2**  One can also define the completion by defining the relevant norms locally and summing over a partition of unity in a fixed locally finite atlas of $M$. In practice, especially for the study of convergence of maps in $W^{k,p}$ topology, one needs to employ the techniques of a-priori estimates for which this local definition will be more practical.

Now let $u \in \mathcal{C}^\infty(\Sigma, M) \subset W^{k,p}(\Sigma, M)$. Consider the set $W^{k,p}(u^*TM)$ defined by

$$W^{k,p}(u^*TM) = W^{k,p}\text{-completion of } \mathcal{C}^\infty(u^*TM).$$

Recall that $\mathcal{C}^\infty(u^*TM)$ is defined to be the set

$$\mathcal{C}^\infty(u^*TM) = \{\xi: \Sigma \to TM \mid \pi \circ \xi = u, \xi \text{ smooth}\},$$
where $\pi : TM \to M$ is the obvious projection. We also denote by $\pi : u^*TM \to \Sigma$ the induced projection $u^*TM|_z = T_{u(z)}M \to z$ with $z \in \Sigma$.

We denote by $\mathcal{F}^{k,p} = W^{k,p}(\Sigma, M)$ the $W^{k,p}$-completion of $\mathcal{F}$.

**Proposition 10.2.3** Assume $M$ is a finite-dimensional closed manifold. Let $k - 2/p > 0$. Then $\mathcal{F}^{k,p}$ carries a $C^\infty$ Banach manifold structure such that its tangent space $T_u\mathcal{F}^{k,p}$ at $u$ is given by

$$T_u\mathcal{F}^{k,p} = W^{k,p}(u^*TM).$$

(10.2.3)

**Proof** We need to provide an atlas for $\mathcal{F}^{k,p}$. Fix a metric $g$ on $M$, and let $\delta = \iota(M, g)$ be the injectivity radius of $(M, g)$. Denote by $D_\delta(TM)$ the subset $D_\delta(TM) = \{(x, v) \in TM \mid |v| < \delta\}$. Then the associated exponential map $\text{Exp} : D_\delta(M) \subset TM \to M \times M$ is well defined and is a diffeomorphism onto its image. We denote $\text{Exp}(x, v) = (x, \exp_x(v))$. Consider the open subset $U_u = \tilde{\exp}_u V_u$ with smooth center $u$, where

$$V_u = \{\xi \in W^{k,p}(u^*TM) \mid |\xi(z)| < \iota(M, g)\}$$

and $\tilde{\exp}_u : V_u \to U_u$ is defined by the formula

$$\tilde{\exp}_u(\xi)(z) := \exp_{u(z)}\xi(z)$$

for $z \in \Sigma$. Then $\tilde{\exp}_u$ becomes a homeomorphism onto its image, which contains $u$ and hence $(U_u, \tilde{\exp}_u^{-1})$ defines a local chart of $W^{k,p}(\Sigma, M)$ modeled by the Banach space $W^{k,p}(u^*TM)$.

Apply this to a countable dense subset of $u$s that lie in $C^\infty(\Sigma, M) \subset W^{k,p}(\Sigma, M)$. By the density of embedding $C^\infty(\Sigma, M) \hookrightarrow W^{k,p}(\Sigma, M)$, it follows that

$$W^{k,p}(\Sigma, M) = \bigcup_{u \in C^\infty(\Sigma, M)} U_u.$$

Next we prove the compatibility of these local charts. □

**Lemma 10.2.4** The collections

$$\{\tilde{\exp}_u^{-1} : U_u \to W^{k,p}(u^*TM) \mid u \in C^\infty(\Sigma, M)\}$$

form a $C^\infty$ atlas of $\mathcal{F}^{k,p}$.

**Proof** We need to prove that the transition map

$$\tilde{\exp}_{u_1}^{-1} \circ \tilde{\exp}_{u_2} : \tilde{\exp}_{u_2}^{-1}(U_{u_2} \cap U_{u_1}) \to \tilde{\exp}_{u_1}^{-1}(U_{u_2} \cap U_{u_1})$$

is $C^\ell$ for all $\ell \in \mathbb{N}$ for any choice of $u_1, u_2 \in C^\infty(\Sigma, M)$. 
From the definition of $\tilde{\exp}$, we have
\[
\tilde{\exp}^{-1}_{u_1} \circ \tilde{\exp}^{-1}_{u_2}(\xi)(z) = \exp^{-1}_{u_1(z)} \circ \exp_{u_2(z)}(\xi(z)).
\]
Note that, since we have chosen $u_1$, $u_2$ to be smooth, the map $z \to \tilde{\exp}^{-1}_{u_1} \circ \tilde{\exp}_{u_2}(\xi(z))$ lies in $W^{k,p}$ if and only if $\xi$ is in $W^{k,p}$ and hence defines a $W^{k,p}$ section of $u_1^*TM$. In addition, it again follows from the smoothness of $u_1$, $u_2$ that the assignment $\xi \to \tilde{\exp}^{-1}_{u_1} \circ \tilde{\exp}_{u_2}$ defines a map
\[
\mathcal{V}_{u_1} \cap \tilde{\exp}^{-1}_{u_1} \circ \tilde{\exp}_{u_2}(\mathcal{V}_2) \to \mathcal{V}_{u_2} \cap \tilde{\exp}^{-1}_{u_2} \circ \tilde{\exp}_{u_1}(\mathcal{V}_1)
\]
that is differentiable infinitely many times. By changing the role of $u_1$ and $u_2$, we prove that the inverse map $\tilde{\exp}^{-1}_{u_2} \circ \tilde{\exp}_{u_1}$ is also differentiable infinitely many times. Hence we have shown that the transition maps are $C^\infty$. See Figure 10.1.

\[
\square
\]

Exercise 10.2.5 Prove that the transition map (10.2.4) is differentiable infinitely many times and also prove the statement on the tangent space.

This finishes the proof.

Remark 10.2.6 We remark that the model Banach space $W^{k,p}(u_1^*TM)$ varies depending on the choice of base points $u$, but they are homeomorphic whenever $u_1$, $u_2$ are sufficiently $C^\infty$-close to each other. Therefore we can compose another homeomorphism of $W^{k,p}(u_2^*TM)$ to $W^{k,p}(u_1^*TM)$ to provide an atlas of $F^{k,p}$ out of the atlas given in Proposition 10.2.3 such that the transition maps in
the modified atlas are differentiable infinitely many times, and hence a Banach manifold structure in the sense of Definition 10.1.5.

From now on, we suppress the super-indices $k, p$ from our notation, unless they are absolutely necessary for clarity.

We regard the assignment $u \mapsto du$ as a section of some (infinite-dimensional) vector bundle, which we now explain. Denote

$$\mathcal{H}_u = \Gamma(\Lambda^1(u^*TM)); \quad \mathcal{H} = \bigcup_{u \in \mathcal{F}} \mathcal{H}_u.$$ 

Then the map $d : u \mapsto du$ defines a section of the bundle $\mathcal{H} \to \mathcal{F}$. This section is not a Fredholm section, though: note that the rank of the vector bundle $u^*TM$ is $2n$ and that of $T\Sigma \otimes u^*TM$ is $4n$. Following standard notation, we denote

$$\Gamma(u^*TM) = \Omega^0(u^*TM), \quad \Gamma(\Lambda^1(u^*TM)) = \Omega^1(u^*TM).$$

Next we involve the given complex structure $j$ on $\Sigma$ and the almost-complex structure $J$ on $(M, \omega)$. The pair $(j, J)$ induces a decomposition

$$\text{Hom}(T\Sigma, u^*TM) = \text{Hom}'_{(j,J)}(T\Sigma, u^*TM) \oplus \text{Hom}''_{(j,J)}(T\Sigma, u^*TM)$$

into a complex linear part and an anti-complex linear part as

$$A = \frac{A - J \cdot A \cdot j}{2} + \frac{A + J \cdot A \cdot j}{2} = A' + A''.$$ 

Equivalently, regarding $A$ as an element in $\Lambda^1(u^*TM)$, we also decompose $A$ into

$$A = A^{(1,0)} + A^{(0,1)}$$

with respect to the almost-complex structures $j, J$. This induces the corresponding decomposition of one-forms with values in $u^*TM$,

$$\Omega^1(u^*TM) = \Omega_{(j,J)}^{1,0}(u^*TM) \oplus \Omega_{(j,J)}^{0,1}(u^*TM),$$

and so

$$\mathcal{H} = \mathcal{H}^{(0,1)} \oplus \mathcal{H}^{(0,1)}$$

with

$$\mathcal{H}_{(j,J)}^{1,0} = \Omega_{(j,J)}^{1,0}(u^*TM), \quad \mathcal{H}_{(j,J)}^{0,1} = \Omega_{(j,J)}^{0,1}(u^*TM).$$

Finally, we define the section

$$\overline{\delta}_{(j,J)} : \mathcal{F} \to \mathcal{H}_{(j,J)}^{0,1} := \mathcal{H}_{j,J}^{0,1}(\cdot)^*TM$$
Fredholm theory

by

\[ u \mapsto \tilde{\partial}_{(j,J)} u := \frac{du + J \cdot du \cdot j}{2}. \]

By definition, a \((j,J)\)-holomorphic map from \((\Sigma, j)\) to \((M, J)\) is a zero of the section \(\tilde{\partial}_{(j,J)}\). We will suppress \(j\) on \(\Sigma\) henceforth whenever there is no danger of confusion. We would like to point out that

\[ \text{rank } u^*TM = \text{rank } \Lambda^{(0,1)}_{(j,J)}(u^*TM)(= 2n), \]

which reflects the two-dimensionality of \(\Sigma\). Whenever there is no danger of confusion, we will suppress \((j,J)\) from the notation.

10.3 Linearizations of \(\tilde{\partial}_{(j,J)}\)

Let \(\Sigma\) be a compact surface of genus \(g\), and let \(J_\Sigma\) be the set of complex structures \(j\) on \(\Sigma\) as before. First we fix a complex structure \(j\) on \(\Sigma\).

**Proposition 10.3.1** Let \(J\) be any compatible almost-complex structure on \((M, \omega)\). Then the assignment

\[ \tilde{\partial}_{(j,J)} : F \to H^{(0,1)}_{(j,J)} \]

defines a Fredholm section if we give \(W^k,p\) topology on \(T_uF\) and \(W^{k-1,p}\) topology on \(H^{(0,1)}_{(j,J,u)}\) for \(k - 2/p > 0\).

In other words, the linearization map

\[ D_u\tilde{\partial}_{(j,J)} : T_uF \to H^{(0,1)}_{(j,J,u)} \]

is a Fredholm operator. The rest of this section will be occupied with the proof of this proposition and the calculation of the index of this Fredholm operator.

Recall

\[ T_uF = \Omega^0(u^*TM), \quad H^{(0,1)}_{(j,J,u)} = \Omega^{(0,1)}_{(j,J)}(u^*TM). \]

We will compute \(D_u\tilde{\partial}_{(j,J)}\) explicitly for that purpose. For the computation, we will use an almost-Hermitian connection (e.g., the canonical connection) of the almost-Kähler structure \((M, \omega, J)\), i.e., the one satisfying

\[ \nabla \omega = 0 = \nabla J. \]
This connection, whose torsion need not be zero, is not the same as the Levi-Civita connection of $g_J$ in general unless $J$ is integrable.

Suppose $d(x_0, x_1) < \text{inj}(M, \nabla)$ so that there exists a unique short geodesic from $x_0$ to $x_1$ with respect to the connection $\nabla$. Then we denote by $\Pi_{x_0}^{x_1} : T_{x_0}M \to T_{x_1}M$ the parallel transport along the unique short geodesic.

We start by finding an explicit expression for the (covariant) linearization of the section $d$. The connection $\nabla$ on $TM$ also provides a connection $\nabla$ on the vector bundle $\mathcal{H} \to \mathcal{F}$ via the parallel translation along the short geodesics as follows.

For a vector field $X$ on $M$ and for a vector $v \in T_{x_0}M$, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be the unique geodesic with $\gamma(0) = x_0$ and $\gamma'(0) = v$. Then we have the formula
\[
\nabla_v X = \frac{d}{ds}igr|_{s=0} \left( \Pi_{x_0}^{x(s)} \right)^{-1}(X(\gamma(s))) \in T_{x_0}M.
\]

We denote by $\nabla_{du}$ the pull-back connection of $\nabla$ on the vector bundle $u^*TM \to \Sigma$ for a map $u : \Sigma \to M$.

Given any section $\Upsilon : \mathcal{F} \to \mathcal{H}$ (say $du$), and $\xi \in T_\xi \mathcal{F}$, we define the covariant derivative by its evaluation against $\xi$ at $z$ given by
\[
\nabla_{du} \Upsilon(\xi)(z) = \frac{d}{ds}igr|_{s=0} \left( \Pi_{u(z)}^{u(\epsilon)} \right)^{-1}(\Upsilon(u_\epsilon^z(\xi)))
\]
for $z \in \Sigma$ with $u_\epsilon^z(\xi) := \exp_{u(z)}(\epsilon\xi(z))$. We will explicitly calculate the covariant linearization $D_{u^*d} : T_{u^*\mathcal{F}} \to \mathcal{H}_u$. Recall that $du$ is a one-form on $\Sigma$ with values in $u^*TM$, and hence we try to evaluate $D_{u^*d}(\xi)(a)$ for $a \in T_\Sigma$.

First, we consider a curve $\lambda : (-\varepsilon, \varepsilon) \to \Sigma$ such that $\lambda(0) = z$, $\lambda'(0) = a$, and consider a one-parameter family of curves
\[
\Gamma : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \to M; \quad (t, s) \mapsto \exp_{\lambda(t)}(s\xi).
\]
Note that $\Gamma(0, s) = \exp_{\gamma}(s\xi) = u_\xi^s(z)$, $\Gamma(t, 0) = u(\lambda(t))$. Therefore we have
\[
\frac{\partial \Gamma}{\partial t}igr|_{t=0, s=0} = du(a), \quad \frac{\partial \Gamma}{\partial s}igr|_{s=0, t=0} = \xi(u(z)).
\]

We recall the following torsion formula:
\[
\nabla_s \frac{\partial \Gamma}{\partial s}igr|_{(t,s)=(0,0)} - \nabla_t \frac{\partial \Gamma}{\partial s}igr|_{(t,s)=(0,0)} = T\left( \frac{\partial \Gamma}{\partial s}igr|_{(s,t)=(0,0)} , \frac{\partial \Gamma}{\partial t}igr|_{(s,t)=(0,0)} \right)
= T(\xi, du(a)).
\]
Now we calculate \( D_u(d)(\xi)(a) \):

\[
D_u(d)(\xi)(a) = \frac{D}{\partial s} \bigg|_{(s,t)=(0,0)} d(u^t) \left( \frac{dy}{dt} \right) = \frac{D}{\partial s} \frac{\partial \Gamma}{\partial t} \bigg|_{(s,t)=(0,0)} + T(\xi, du(a))
\]

\[
= \frac{D}{\partial t} \bigg|_{t=0} \left( \xi + T(\xi, du(a)) \right) = \nabla_{du(a)}\xi + T(\xi, du(a)).
\]

Now we denote by \( \widetilde{T}_{du} \) the one-form on \( \Sigma \) with values in \( \text{End}(u^*TM) \) defined by

\[
\widetilde{T}_{du}(\xi)(a) = T(\xi, du(a)) \quad \text{for } a \in T\Sigma, \xi \in u^*TM.
\]

With these notations, we have obtained the formula for the (covariant) linearization of \( d \),

\[
D_u d = \nabla_{du} \+ \widetilde{T}_{du}. \quad (10.3.5)
\]

Then, by the almost-Kähler property of the connection \( \nabla \), it follows that

\[
D_u \partial_{(j,J)} = (\nabla_{du} + \widetilde{T}_{du})_{(0,1)}^{(0,1)}.
\]

More explicitly, we can write this as

\[
D_u \partial_{(j,J)}(\xi)(a) = \frac{1}{2} \left( \nabla_{du(a)}\xi + J \cdot \nabla_{du(j,a)}\xi \right) + \frac{1}{2} \left( T(\xi, du(a)) + J \cdot T(\xi, du(j,a)) \right). \quad (10.3.6)
\]

We summarize the above calculation in the following proposition.

**Proposition 10.3.2** Let \( \nabla \) be an almost-Hermitian connection of \((M, \omega, J)\). Fix \( j \) on \( \Sigma \) and consider the \( \partial_{(j,J)} \) as a section of the bundle \( \mathcal{H}^{(0,1)}_{(j,J)} \rightarrow F \). Then the covariant linearization

\[
D_u \partial_{(j,J)} : \Omega^0(u^*TM) \rightarrow \Omega^{(0,1)}_{(j,J)}(u^*TM)
\]

of the section \( \partial_{(j,J)} \) at a smooth map \( u : \Sigma \rightarrow M \) is a differential operator of order 1 that is elliptic.

**Proof** It follows from (10.3.6) that \( D_u \partial_{(j,J)} \) is a first-order differential operator and its symbol is given by the function

\[
\sigma \left( D_u \partial_{(j,J)} \right) : T^* \Sigma \setminus \sigma_{T^* \Sigma} \rightarrow \text{Hom}_{\mathbb{R}}(\pi^*(u^*TM), \pi^*(\Lambda^1(u^*TM)));
\]

\[
\alpha_z \mapsto \frac{\alpha_z \otimes \text{Id} + \alpha_z \circ j(z) \otimes J(u(z))}{2}.
\]
One can easily check from this expression for the image of $\alpha_z$ that it is an isomorphism for any $\alpha_z \neq 0$. This proves that the operator is elliptic.

We would like to remark that, although the expression for $D_u \partial_{(j,J)}$ looks like it depends on the choice of connection, it will not depend on the choice of connection when $u$ is $(j,J)$-holomorphic, i.e., $\partial_{(j,J)} u = 0$.

This, together with the Riemann–Roch formula (or by the Atiyah–Singer index formula), immediately gives rise to the following theorem.

**Theorem 10.3.3** Assume $(\Sigma, j)$ is a closed Riemann surface with genus $g$ and $[u] = \beta \in H_2(M, \mathbb{Z})$. Then

$$D_u \partial_{(j,J)} : T_u F^{(k,p)} \to \mathcal{H}^{k-1,p}_{(j,J)}$$

is a Fredholm operator with its index given by

$$\text{Index } D_u \partial_{(j,J)} = 2c_1(\beta) + n\chi(\Sigma) = 2c_1(\beta) + 2n(1 - g).$$

Hence $\partial_{(j,J)}$ is a Fredholm section of the Banach bundle $\mathcal{H}^{k-1,p}_{(j,J)} \to F^{(k,p)}$.

Now we fix a homology class $\beta \in H_2(M, \mathbb{Z})$. We consider a subset of $F^{k,p}$

$$F^{k,p}(\beta) = \{ u \in F^{k,p} \mid [u] = \beta \}$$

and the zero set

$$\tilde{M}^{k,p}(J; \beta) = \{ u \in F^{k,p}(\beta) \mid \partial_{(j,J)} u = 0 \}$$

for $k \geq 1$, $p > 2$. By elliptic regularity, any element of $\tilde{M}^{k,p}(J; \beta)$ is smooth and so the set $\tilde{M}^{k,p}(J; \beta)$ does not depend on $k$, $p$. We denote the common space by $\tilde{M}(J; \beta)$, regarding it as a subset of $C^\infty(\Sigma, M)$. Again by elliptic regularity and the boot-strap argument, the induced $W^{k,p}$ topologies on $\tilde{M}(J; \beta)$ are all equivalent to the smooth topology thereon induced from $C^\infty(\Sigma, M)$.

## 10.4 Mapping transversality and linearization of $\bar{\partial}$

Now we hope to prove that $\tilde{M}(J; \beta)$ carries the structure of a smooth manifold. We wish to obtain such a transversality for a good choice of almost-complex structures $J \in \mathcal{J}_\omega$.

For this purpose, we consider $J$ as a parameter and $\mathcal{J}_\omega$ as a parameter space which is the space of $\omega$-compatible almost-complex structures. For the rest of the section, we will fix $j$ and so suppress the dependence on $j$. For the technical reason that $\mathcal{J}_\omega$ with $C^\infty$ topology is not a Banach manifold, we also need to
consider its $C^\ell$ analog, which we denote by $\mathcal{F}_\omega^\ell$ with $\ell \geq 1$. Then $\mathcal{F}_\omega^\ell$ carries a Banach manifold structure.

For given $(k, p)$ and $\ell$ and class $\beta \in \pi_2(M)$, we consider the bundle $\mathcal{H}''(\beta) \to \mathcal{F}^{k,p}(\beta) \times \mathcal{F}_\omega^\ell$ whose fiber at $(u, J)$ is given by $\mathcal{H}^{(0,1)}_{(u,J)}$, i.e.,

$$\mathcal{H}''(\beta) = \bigcup_{(u,J) \in \mathcal{F}^{k,p}(\beta) \times \mathcal{F}_\omega^\ell} \{ (u, J) \times \mathcal{H}^{(0,1)}_{(u,J)} \}.$$

Then we regard the map $\overline{\partial} : \mathcal{F}^{k,p}(\beta) \times \mathcal{F}_\omega^\ell \to \mathcal{H}''(\beta)$ as a section defined by $\overline{\partial}(u, J) = \overline{\partial}_{J}u$.

We will study the linearization of $\overline{\partial}$ at each $(u, J) \in \overline{\partial}^{-1}(0)$. We denote the zero set $\overline{\partial}^{-1}(0)$ by

$$\overline{\mathcal{M}}(\beta) = \left\{ (u, J) \in \mathcal{F}^{k,p}(\beta) \times \mathcal{F}_\omega^\ell \mid \overline{\partial}_{(j,J)}u = 0 \right\},$$

which we call the universal moduli space. Denote by $\pi_2 : \mathcal{F}^{k,p}(\beta) \times \mathcal{F}_\omega^\ell \to \mathcal{F}_\omega^\ell$ the projection. Then we have $\overline{\mathcal{M}}(J; \beta) = \pi_2^{-1}(J) \cap \overline{\mathcal{M}}(\beta)$.

The following theorem summarizes the main scheme of the study of the moduli problem of pseudoholomorphic curves.

**Theorem 10.4.1** Let $0 < \ell < k - 2/p$. Denote by $\overline{\mathcal{M}}^{\text{inj}}(\beta)$ the subset

$$\overline{\mathcal{M}}^{\text{inj}}(\beta) = \{ (u, J) \in \overline{\mathcal{M}}(\beta) \mid u \text{ is somewhere injective} \}.$$

(1) $\overline{\mathcal{M}}^{\text{inj}}(\beta)$ is an infinite-dimensional $C^\ell$ Banach manifold.

(2) The projection $\Pi_{\beta} = (\pi_2)|_{\overline{\mathcal{M}}^{\text{inj}}(\beta)} : \overline{\mathcal{M}}^{\text{inj}}(\beta) \to \mathcal{F}_\omega^\ell$ is a Fredholm map and its index is the same as that of $D\overline{\partial}_J(u)$ for a particular (and hence any) $u \in \overline{\mathcal{M}}^{\text{inj}}(\beta)$.

The rest of the section will be occupied by the proof of this theorem. For the first statement, it suffices to prove that the covariant linearization

$$D\overline{\partial}(u, J) : T_u\mathcal{F}^{k,p}(\beta) \oplus T_J\mathcal{F}_\omega^\ell \to \mathcal{H}''_{(u,J)}$$

is surjective at $(u, J)$ for which $u$ is a somewhere-injective curve. To avoid confusion arising from the notation $\overline{\partial}$, we denote $\Phi = \overline{\partial}$ in the calculation below.

Note that $D\Phi(u, J) = D_1\Phi(u, J) + D_2\Phi(u, J)$, where

$$D_1\Phi(u, J) : T_u\mathcal{F}^{k,p}(\beta) \to \mathcal{H}''_{(u,J)},$$

$$D_2\Phi(u, J) : T_J\mathcal{F}_\omega^\ell \to \mathcal{H}'_{(u,J)}.$$
are the associated partial derivatives. A straightforward calculation shows that

\[ D_2 \Phi(u, J)(B) = \frac{1}{2} B \cdot du \cdot j. \]  

(10.4.7)

Obviously, by definition, we have \( D_1 \Phi(u, J) = D_u \hat{\partial}(j, J) \). We have already computed \( D_u \hat{\partial}(j, J) \) and showed that it is formally elliptic. Hence its \( W^{k, p} \)-extension, again denoted by \( D_1 \Phi(u, J) \), is Fredholm and in particular \( \text{coker}(D_1 \Phi(u, J)) \) is finite-dimensional. This implies that \( \text{Im} \, D \Phi(u, J) \) itself is a closed subspace of \( \mathcal{H}_{(u, J)}'' \) with finite codimension since \( \mathcal{H}_{(u, J)}'' \) contains \( \text{Im} \, D \Phi(u, J) \).

Therefore it will suffice to prove that \( \text{coker} \, D \Phi(u, J) = \{ 0 \} \) in order to prove the surjectivity of \( D \Phi(u, J) \). The Hahn–Banach theorem enables us to reduce its proof to that of the statement that, for any continuous linear functional \( \eta : \mathcal{H}_{(u, J)}'' \to \mathbb{R} \) such that \( \eta|_{\text{Im} \, D \Phi(u, J)} = 0 \), we have \( \eta = 0 \). This will follow from the stronger lemma below.

**Lemma 10.4.2** If any continuous linear functional \( \eta \) on \( L^p \left( \Lambda_{j,J}^1 (u^*TM) \right) \) vanishes on \( D \Phi(u, J)(C^\infty (\Gamma(u^*TM))) \), then \( \eta = 0 \).

Postponing this lemma for a while, we finish the proof of surjectivity. Let \( k, \rho \) be any constants given as above. Regard \( \mathcal{H}_{(u, J)}''' \) as a subspace of \( L^p \) via the continuous embedding \( \mathcal{H}_{(u, J)}''' = W^{k-1, \rho} \left( \Lambda_{j,J}^1 (u^*TM) \right) \hookrightarrow L^p \left( \Lambda_{j,J}^1 (u^*TM) \right) \).

Let \( \eta \) be a continuous linear functional on \( W^{k-1, \rho} \left( \Lambda_{j,J}^1 (u^*TM) \right) \) that vanishes on \( \text{Im} \, D \Phi(u, J) \). Since \( \text{Im} \, D \Phi(u, J) \supset D \Phi(u, J)(C^\infty (\Gamma(u^*TM))) \), it vanishes on \( D \Phi(u, J)(C^\infty (\Gamma(u^*TM))) \). By Lemma 10.4.2, we obtain \( \eta \equiv 0 \) on \( L^p \) and so on \( W^{k-1, \rho} \). This finishes the proof of surjectivity modulo the proof of Lemma 10.4.2.

The rest of the section will be occupied with the proof of Lemma 10.4.2.

**Proof of Lemma 10.4.2** We identify

\[ (L^p(\Lambda_{0,1}(u^*TM)))^* \cong L^q(\Lambda_{1,0}(u^*TM)) \]

with \( 1/p + 1/q = 1, 1 < q < 2 \), via the integral pairing on \( \Sigma \)

\[ L^p(\Lambda_{0,1}(u^*TM)) \times L^q(\Lambda_{1,0}(u^*TM)) \to \mathbb{R}; \quad (f, g) \mapsto \int_\Sigma \langle f, g \rangle, \]

which is nondegenerate. We will prove that \( \eta \equiv 0 \) holds whenever \( \eta \in L^q \left( \Lambda_{j,J}^{(1,0)} (u^*TM) \right) \) satisfies

\[ \int_\Sigma \langle D \Phi(u, J)(\xi, B), \eta \rangle = 0 \]  

(10.4.8)

for all \( \xi \in C^\infty (u^*TM) \) and \( B \in C^\ell \).
Equation (10.4.8) can be decomposed into
\[ \int_{\Sigma} \langle D_1 \Phi(u, J) \xi, \eta \rangle = 0, \quad (10.4.9) \]
\[ \int_{\Sigma} \langle D_2 \Phi(u, J) B, \eta \rangle = 0 \quad (10.4.10) \]
for all \( \xi \in T_u \mathcal{F}(A) \) and \( B \in C^\ell \). The first equation is equivalent to
\[ (D_1 \Phi(u, J))^{\dagger} \eta = 0 \]
in the distribution sense, where \((D_1 \Phi(u, J))^{\dagger}\) is the adjoint of \(D_1 \Phi(u, J)\).

Note that \((D_1 \Phi(u, J))^{\dagger}\) is a first-order formally elliptic differential operator with \(C^{\ell-1}\)-coefficients and its principal symbol is given by
\[ \sigma(D_u \partial_{(j, l)})^{\dagger} : T^* \Sigma \setminus \sigma_{T^* \Sigma} \to \text{Hom}_{\mathbb{R}}(\pi^* (\Lambda^{(0,1)}(u^* TM))^{\dagger}, \pi^* (u^* TM)^{\dagger}), \]
where \(\pi : T^* \Sigma \to \Sigma\) is the projection and \((\cdot)^{\dagger}\) is the dual space. It is easy to check that \(\sigma(D_u \partial_{(j, l)})^{\dagger}(\alpha)\) is an isomorphism of a (complex) scalar type since \(\sigma(D_u \partial_{(j, l)})\) is one.

Then general elliptic theory (see Proposition 10.1.13) implies that \(\eta\) must be differentiable and \(\eta\) is a classical solution of the equation
\[ (D_1 \Phi(u, J))^{\dagger} \eta = 0. \quad (10.4.11) \]

The following unique continuation lemma is a consequence of Aronszajn’s unique continuation lemma (Aro57) for an elliptic operator of Laplacian type, which arises by considering the second-order equation
\[ (D_1 \Phi(u, J))(D_1 \Phi(u, J))^{\dagger} \eta = 0. \]

**Lemma 10.4.3** (Unique continuation lemma) *Any solution of* (10.4.11) *vanishes identically if it vanishes on some open set of* \(\Sigma\).*

**Exercise 10.4.4** Prove the above uniqueness lemma by considering the perturbed Cauchy–Riemann equation \(\bar{\partial} \eta + A(z) \eta = 0\) on an open domain \(D \subset \mathbb{C}\) for \(\eta : D \to \mathbb{C}^n\), where \(A : D \to \text{End}_\mathbb{R}(\mathbb{C}^n)\) is a smooth function. In fact, it suffices to assume that \(A\) is continuous. (See (FHS95) for the proof.)

Therefore, we need only prove that \(\eta \equiv 0\) on some open set. *Here enters the condition of somewhere injectiveness* and the second equation
\[ \int_{\Sigma} \langle D_2 \Phi(u, J) B, \eta \rangle = 0. \quad (10.4.12) \]
Since \(u\) is assumed to be somewhere injective, there exists \(z_0 \in \Sigma\) such that
\[ du(z_0) \neq 0, \quad \#(u^{-1}(u(z_0))) = 1. \quad (10.4.13) \]
It follows that there exists an open neighborhood \( z_0 \in V \subset \Sigma \) such that (10.4.13) holds with \( z_0 \) replaced by \( z \) for all \( z \in V \).

Now we need to examine the tangent space \( T_J J \omega \). Recall that \( J \in J \omega \) is a section of \( \text{End}(TM) \) such that \( J^2 = -\text{id}, \quad \omega(J \cdot, J \cdot) = \omega(\cdot, \cdot) \).

An element \( B \in T_J J \omega \) is a section of \( \text{End}(TM) \) that satisfies
\[
J B + B J = 0, \quad \omega(B \cdot, J \cdot) + \omega(J \cdot, B \cdot) = 0.
\]
The second equation here is nothing but the statement that \( B \) is symmetric with respect to the metric \( g_J = \omega(\cdot, J \cdot) \). Formally, we consider the fiber bundle
\[
\bigcup_{x \in M} S_\omega(x) = S_\omega \rightarrow M,
\]
where \( S_\omega(x) \) is defined to be the set
\[
\{ B_x \in \text{End}(T_x M) \mid J(x)B_x + B_xJ(x) = 0, \omega_x(B_x, J(x)) + \omega_x(J(x), B_x) = 0 \}.
\]
Then \( B \in T_J J \omega \) defines a section of the bundle \( S_\omega \). Note that \( S_\omega(x) \) is isomorphic to the vector space \( S(\mathbb{R}^{2n}) \):
\[
S(\mathbb{R}^{2n}) = \{ a \in \text{End}(\mathbb{R}^{2n}) \mid aJ_0 + J_0a = 0, a \text{ is symmetric} \},
\]
where \( S(\mathbb{R}^{2n}) \) is \( S_\omega(x) \) for \( J_0 : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \), which is a standard complex structure, and \( g_J = \langle \cdot, \cdot \rangle \) is the standard Euclidean inner product. From this, we easily obtain \( \dim S(\mathbb{R}^{2n}) = n^2 + n \).

**Lemma 10.4.5** Let \( e \in \mathbb{R}^{2n} \) be a non-zero fixed vector; then \( S(\mathbb{R}^{2n}) \cdot e = \mathbb{R}^{2n} \), i.e., any vector in \( \mathbb{R}^{2n} \) has the form \( a \cdot e \) for some \( a \in S(\mathbb{R}^{2n}) \subset \text{End}(\mathbb{R}^{2n}) \).

**Exercise 10.4.6** Prove this lemma.

We will prove that \( \eta \equiv 0 \) on \( V \) by contradiction, shrinking \( V \) if necessary.

Suppose that there exists \( z_1 \in V \) such that \( \eta(z_1) \neq 0 \). Since the rank of \( du(z_1) \) is either 0 or 2 if \( \partial_{(j,j)} u(z_1) = 0 \) (why?), \( du(z_1) \) must be an injective map if \( du(z_1) \neq 0 \). Using the fact that the set of critical points of a non-constant \( J \)-holomorphic map \( u \) is isolated, we may also assume that, on slightly perturbing the given point \( z_1 \) inside \( V \), \( du(z_1) \neq 0 \) and hence \( du(z_1) \) is an injective map. We choose a unit vector \( v_{z_1} \in T_{z_1} \Sigma \) such that \( \eta(z_1)(v_{z_1}) \neq 0 \). Then
\[
\eta(z_1)(jv_{z_1}) \neq 0.
\]
Then, using the above lemma, we can find \( a_x \in S_\omega(x) \) such that
\[
a_x \cdot du(z_1)(jv_{z_1}) = \eta(z_1)(v_{z_1})
\]
with \( x = u(z_1) \). Since \( u \) is an embedding on \( V \) and \( z_1 \in V \), there exists an open neighborhood \( U \subset M \) of \( x = u(z_1) \) such that \( u^{-1}(U) = V \).

By the choice of \( v_{z_1} \) and \( a_x \), we have

\[
\langle a_x \cdot du(z_1)(jv_{z_1}), \eta(z_1)(v) \rangle = |\eta(z_1)(v_{z_1})|^2 > 0.
\]

Since \( \{v_{z_1}, jv_{z_1}\} \) form an orthonormal basis of \( T_{z_1} \Sigma \) and \( \eta \) is a (0, 1)-form with respect to \((j, J)\), we derive from the definition of the operator inner product

\[
\langle a_x \cdot du(z_1)(j), \eta(z_1) \rangle := \langle a_x \cdot du(z_1)(j)(v_{z_1}), \eta(z_1)(v_{z_1}) \rangle
\]

\[
+ \langle a_x \cdot du(z_1)(j)(jv_{z_1}), \eta(z_1)(jv_{z_1}) \rangle
\]

\[
= 2\langle a_x \cdot du(z_1)(jv_{z_1}), \eta(z_1)(v_{z_1}) \rangle
\]

\[
= 2|\eta(z_1)(v_{z_1})|^2 > 0.
\]

Here, for the penultimate equality, we use the property that both \( du \) and \( \eta \) are complex linear with respect to \((j, J)\). See Figure 10.2. We choose a smooth local section \( B \in \Gamma(S_{\omega}) \) with \( B(x) = a_x \) and \( \text{supp} \ B \subset U \). If we choose \( U' \subset U \) sufficiently small that \( u(z_1) \in U' \) and \( u^{-1}(\overline{U'}) \subset V \), we will have

\[
\langle B(u(z))du(z) \cdot j_z, \eta(z) \rangle > 0
\]

for all \( z \in u^{-1}(\overline{U'}) \). Then we choose a cut-off function \( \chi : M \to \mathbb{R} \), such that

\[
\chi \equiv 1 \text{ on } U', \quad \text{supp} \xi \subset U.
\]

If we were to choose \( \chi \) so that it is sufficiently \( L^1 \)-close to the characteristic function \( \chi_{U'} \), then we would obtain

\[
0 < \int_{\Sigma} \left\{ \frac{1}{2} (\chi B)(u) \cdot du \circ j, \eta \right\} = \int_{\Sigma} \langle D\Phi_2(u, J)(\chi B), \eta \rangle
\]

by the formula (10.4.7) of \( D\Phi_2(u, J) \). On applying to the variation \( \delta J = \chi B \) the standing hypothesis that \( \int_{\Sigma} \langle D\Phi_2(u, J)\delta J, \eta \rangle = 0 \) holds for all \( \delta J \), this contradicts the hypothesis. This proves the surjectivity of \( D\Phi(u, J) \).

We go back to the proof of the theorem. Once we have proved the surjectivity of \( D\Phi(u, J) \), an immediate consequence of the implicit function theorem is that \( \widehat{M}(\beta) \) is an (infinite-dimensional) smooth submanifold of \( \mathcal{F}^{k,p}(A) \times \mathcal{F}^{k,p}(\mathbb{R}) \times \mathcal{F}^{k,p}(\mathbb{C}) \).
10.4 Mapping transversality and linearization of $\bar{\partial}$

$\mathcal{J}^\ell_\omega$ near the given pair $(u, J) \in \tilde{M}^{\text{ini}}(\beta)$. Hence the proof of statement (1) of Theorem 10.4.1 follows.

We now consider a neighborhood of $(u, J)$ in $\tilde{M}^{\text{ini}}(\beta) \subset \mathcal{F}^{k,p}(\beta) \times \mathcal{J}^\ell_\omega$ provided in the theorem, and the projection map

$$\Pi_\beta := \pi_\Sigma|_{\tilde{M}^{\text{ini}}(\beta)} : \mathcal{J}_\omega; \quad \Pi_\beta(u, J) = J.$$  

We denote by $\iota(g; \beta)$ the topological index associated with $(j, u) \in \mathcal{J}(\Sigma) \times \mathcal{F}_g(M, \omega; \beta)$ with $g_\Sigma = g$. This topological index, which coincides with the analytical index $\text{Index} D_{(j,u)\bar{\partial} J}$ for any $J \in \mathcal{J}_\omega$, can be computed to be

$$\text{Index} D_{\bar{\partial}(j,J)}(u) = \iota(g; \beta)$$

with

$$\iota(g; \beta) = \begin{cases} 
2(c_1(M, \omega)(\beta) + n(1 - g)) & \text{for } g \geq 2, \\
2(c_1(M, \omega)(\beta) + 1) & \text{for } g = 1, \\
2(c_1(M, \omega)(\beta) + n) & \text{for } g = 0 
\end{cases} \quad (10.4.14)$$

by the Riemann–Roch formula.

Statement (2) of Theorem 10.4.1 can be rephrased in the following way.

**Proposition 10.4.7** $\Pi_\beta : \tilde{M}(\beta) \to \mathcal{J}_\omega$ is a Fredholm map with index

$$\text{Index} D\Pi_\beta(u, J) = \text{Index} D_{\bar{\partial}(j,J)}(u).$$

**See Figure 10.3.**

This, together with the Sard–Smale theorem, immediately gives rise to the following corollary.
**Corollary 10.4.8**  For a dense set of $J^\ell_{\text{reg}} \subset J^\ell_\omega$, the set $\widetilde{M}^{(j)}(J; \beta)$ is a smooth manifold of dimension $\iota(g; \beta)$ for $J \in J^\ell_{\text{reg}}$.

**Proof of Proposition 10.4.7**  Note that $D\Pi_\beta : T_{(u, J)}\widetilde{M}(\beta) \to T_JJ_\omega$ is just the restriction of a projection: we have $D\Pi_\beta(\xi, B) = B$ for $(\xi, B) \in T_{(u, J)}\widetilde{M}(\beta) = \ker D\Phi(u, J)$.

The kernel $\ker D\Pi_\beta(u, J)$ is of finite dimension, since

$$\ker D\Pi_\beta(u, J) = \ker D\Phi(u, J) \cap \{T_uF \times \{0\}\} = \ker D_1\Phi(u, J) = \ker D\overline{\partial}_{(J, J)}(u).$$

We will show that $\coker D\Pi_\beta(u, J) \simeq \coker(D\overline{\partial}_{J, J}u)$. Consider the following diagram:

$$\begin{array}{ccc}
0 & \longrightarrow & \ker D\Phi \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
T_vF & \oplus & T_JJ_w \\
\downarrow D_1\Phi & & \downarrow D_2\Phi \\
T_JJ_w & & \Omega^{(0,1)}_\omega(u^*TM) \\
\downarrow \pi_2 & & \downarrow \pi_1 \\
& & 0 \\
\end{array}$$

The proof will follow once we prove that there exists a natural isomorphism $\coker D\Pi_\beta \to \coker D\overline{\partial}_{(J, J)}(u)$, i.e., an isomorphism

$$T_JJ_\omega^\ell/\text{Im}(D\Pi_\beta) \simeq \Omega^{(0,1)}_\omega(u^*TM)/\text{Im}(D\overline{\partial}_{(J, J)}(u)).$$

Firstly, well-definedness: we will show that the assignment $T_JJ_\omega^\ell/\text{Im}(D\Pi_\beta) \to \Omega^{(0,1)}_\omega(u^*TM)/\text{Im}(D\overline{\partial}_{(J, J)}(u))$ is well defined. We recall that $D_1\Phi(u, J)\xi = D\overline{\partial}_{(J, J)}(u)\xi$. If $B \in \text{Im}(D\Pi_\beta)$, i.e., if $(\xi, B) \in T_{(u, J)}\widetilde{M} = \ker(D\Phi(u, J))$ for some $\xi \in \Omega^0(u^*TM)$, then we have $D_1\Phi(u, J)\xi + D_2\Phi(u, J)B = 0$ and hence

$$D_2\Phi(u, J)B = -D_1(\Phi(u, J))\xi.$$

But the right-hand side is contained in $\text{Im} D_1\Phi(u, J) = \text{Im} D\overline{\partial}_{(J, J)}(u)$. This proves that the map $[B] \mapsto [D_2\Phi(u, J)B]$ is well defined.
Secondly, injectivity: suppose that $D_2 \Phi(u, J)(B) \in \text{Im}(D_0 \tilde{\partial}_{\ell,J})$, i.e., there exists $\xi \in T_v \mathcal{F}$, such that $D_2 \Phi(u, J)(B) = D_0 \tilde{\partial}_{\ell,J} \xi$. Then we have $D_2 \Phi(u, J)(B) + D_1 \Phi(u, J)(-\xi) = 0$; therefore,

$$(-\xi, B) \in \ker(D\Phi(u, J)) = T_{(u,J)} \tilde{M}$$

and hence $B \in \text{Im} D\Pi_\beta(u, J)$ and hence $[B] = 0$.

Thirdly, surjectivity: again, by the Hahn–Banach theorem, we need only prove that any $\eta$ satisfying

$$\langle \text{Im} D_1 \Phi(u, J), \eta \rangle = 0 = \langle \text{Im} D_2 \Phi(u, J), \eta \rangle$$

must be zero. However, this equation is equivalent to $\langle \text{Im} D\Phi(u, J), \eta \rangle = 0$. Since $D\Phi(u, J)$ is surjective, we have proved that $\eta \equiv 0$. $\Box$

Now we indicate how one can prove the $C^\infty$ version for $J^\ell_\omega$ of Theorem 10.4.1. There are two ways to achieve this goal.

### Floer’s approach

One way is to follow Floer’s original approach of the transversality proof given in (Fl88a). At each given $J_0$, we consider a sequence of positive numbers $\epsilon = \{\epsilon_k\}_{k=0}^\infty$ and define the space, denoted by $\mathcal{U}_\epsilon(J_0)$, of $J$s, which can be written as

$$J = J_0 \exp(-J_0 B),$$

where $B \in \text{End}(TM)$ satisfies the conditions

(i) $\omega(Bv, w) + \omega(v, Bw) = 0$ for all $v, w$,

(ii) $\sum_{k=0}^\infty \epsilon_k |D^k B| < \infty$ and

(iii) $J$ is tame to $\omega$, i.e., $\omega$ is $J$-positive.

Then Floer’s Lemma 5.1 in (Fl88a) proves that, if $\epsilon$ decays sufficiently fast, $\mathcal{U}_\epsilon(J_0)$ becomes a Banach manifold that is dense in an $L^2$ neighborhood of $J_0$ in $\mathcal{F}_\omega$. We then apply the above transversality proof directly, working on $\mathcal{U}_\epsilon(J_0)$ to obtain a residual subset $\mathcal{U}_\epsilon(J_0)_{\text{reg}} \subset \mathcal{U}_\epsilon(J_0)$ such that for any $J \in \mathcal{U}_\epsilon(J_0)_{\text{reg}}$ all somewhere-injective $J$-holomorphic maps are transversal. By varying $J_0 \in \mathcal{F}_\omega$ and taking the union of $\mathcal{U}_\epsilon(J_0)_{\text{reg}}$ over $J_0$, we obtain a subset $\mathcal{F}_\omega^\text{reg}$ of $\mathcal{F}_\omega$ in the second category on which the required transversality holds.

### Taubes’ approach

Another approach is originally credited to Taubes, which we now explain. It has been explained also by McDuff and Salamon in (MSa94, MSa04).
Denote by $\widetilde{M}^{\text{inj}}(M)$ the universal moduli space consisting of the triple $((\Sigma, j), u, J)$ with $u$ being somewhere injective and $J \in C^k$. We take an exhaustive countable union of the moduli space of complex structures $M_g$ on genus-$g$ compact surfaces into

$$M_g = \bigcup_{l=1}^{\infty} \mathcal{U}_{g,l},$$

where $\mathcal{U}_{g,l}$ is compact and $\mathcal{U}_{g,l} \subset \mathcal{U}_{g,l+1}$.

The following is the key definition which enables one to carry out the step-by-step proof at each finite stage of $(G, K; L)$.

**Definition 10.4.9** For each given pair $(G, K; L) \in \mathbb{N}^3$, we choose a sufficiently large $k = k(G, K) \in \mathbb{N}$ and consider the subset $\widetilde{M}^{\text{inj}}_{(G, K; L)}(M) \subset \widetilde{M}^{\text{inj}}(M)$ consisting of the triple $((\Sigma, j), u, J) \in \widetilde{M}^{\text{inj}}(M)$ which satisfies the following criteria:

1. $g_\Sigma \leq G$ and $l \leq L$;
2. $j \in \mathcal{U}_{g,l} \subset M_g$;
3. $u$ satisfies $\omega(|u|) \leq K$ and $\|d u\|_\infty \leq K$;
4. there exists a somewhere-injective point $z \in \Sigma$ such that

$$\inf_{z' \neq z} \frac{\text{dist}(u(z)), u(z')}{\text{dist}(z, z')} > \frac{1}{K}. \quad (10.4.15)$$

Note that the last condition makes sense since $z$ is a somewhere-injective point, which exists by virtue of the hypothesis that $u$ is somewhere injective. Furthermore, any somewhere-injective $J$-holomorphic map satisfies (1)–(4) for some choice of $(G, K; L)$. We denote by $J^{\text{reg}}_{(G, K; L)} \subset J^k_\omega$ the regular values of the projection $\widetilde{M}^{\text{inj}}_{(G, K; L)}(M) \to J^k_\omega$. We also choose $k$ so that $k \to \infty$ as $\min\{G, K, L\} \to \infty$.

The proof of the following proposition is left as an exercise.

**Proposition 10.4.10** For each $(G, K; L) \in \mathbb{N}^3$, choose $k = k(G, K; L)$ as above. We define the subset $J^{\text{reg}}_{(G, K; L)}$ to be the regular values of the projection $\widetilde{M}^{\text{inj}}_{(G, K; L)}(M) \to J^k_\omega$, and

$$J^{\text{reg}}_{(G, K; L)} = J^{\text{reg}}_{(G, K; L)} \cap J_\omega.$$

Then $J^{\text{reg}}_{(G, K; L)}$ is open and dense in $J_\omega$ in $C^\infty$ topology.
Exercise 10.4.11 Give the proof of this proposition.

Now we define $\mathcal{J}_{Ta}^{reg}$ to be the intersection

$$\mathcal{J}_{Ta}^{reg} = \bigcap_{(G,K) \in \mathbb{N}^2} \bigcap_{L \in \mathbb{N}} \mathcal{J}_{(G,K,L)}^{reg}.$$ 

Since each subset $\mathcal{J}_{(G,K,L)}^{reg} \subset \mathcal{J}_{G}^{\omega}$ is open and dense in $C^\infty$ topology, the intersection is of the second category and hence the proof has been obtained.

Remark 10.4.12 The argument used in Taubes’ approach is rather subtle: on the one hand, we should use Banach manifold $\mathcal{J}_{G}^{k}$, rather than $\mathcal{J}_{G}^{\omega}$ with $C^\infty$ topology; and on the other hand, the above transversality proof has to be applied to each of the homology classes of $M$ and genus $g$ of the domain. Unfortunately, the inductive argument over $\mathcal{J}_{g,\beta}^{k,reg}$ as $k \to \infty$ cannot be applied because the selected dense subset $\mathcal{J}_{g,\beta}^{k,reg} \subset \mathcal{J}_{G}^{k,\omega}$ is not an open subset and hence does not carry the Banach manifold structure in general. Furthermore, the intersection $\mathcal{J}_{g,\beta}^{k,reg} \subset \mathcal{J}_{G}^{\omega}$ over $\beta$, $g$ is not necessarily a subset of the second category in $\mathcal{J}_{G}^{\omega}$. Because of this, we cannot construct the required dense subset thereof by directly taking the intersection of $\mathcal{J}_{g,\beta}^{k,reg} \subset \mathcal{J}_{G}^{\omega}$ over $\beta$ and $g$. However, with the subset $\tilde{M}_{(G,K,L)}^{k,\text{inj}}(M)$ defined above, the above proposition guarantees that $\mathcal{J}_{(G,K,L)}^{k,\text{reg}} \cap \mathcal{J}_{G}^{\omega}$ is not only dense but also open in $\mathcal{J}_{G}^{\omega}$, which enables us to take the intersections to obtain the required subset of the second category, $\mathcal{J}_{Ta}^{reg}$.

10.5 Evaluation transversality

In this section we give an important ingredient for applications of pseudoholomorphic curves, namely the evaluation map transversality. This transversality plays a crucial role in the definition of counting invariants such as in the construction of Gromov–Witten invariants and also in the analysis of gluing pseudoholomorphic curves.

It turns out that the rigorous proof of this evaluation transversality is rather subtle. We give a conceptually canonical proof of the evaluation map transversality adapting the approach taken by Le and Ono (LO96) and Zhu and the author (OhZ09) (in their studies of one-jet evaluation transversality) to the current easier context of zero-jet evaluation transversality. This proof is based on a standard structure theorem of distributions with point support.

A different approach to this evaluation transversality is given in (MSa04), which is more geometric. The naturality of our proof has an advantage in
that it can be generalized to higher-jet transversality using essentially the same scheme except for the usage of holomorphic jets of $J$-holomorphic curves. We refer readers to (Oh11a) for this generalization.

We first recall the structure theorem of the distributions with point support from Section 4.5 (GS68), Theorem 6.25 of (Rud73), to whose proof we refer readers.

**Theorem 10.5.1** (Distribution with point support) Suppose that $\psi$ is a distribution on open subset $\Omega \subset \mathbb{R}^n$ with $\text{supp} \, \psi = \{ p \}$ and of finite order $N < \infty$. Then $\psi$ has the form

$$\psi = \sum_{|\alpha| \leq N} D^\alpha \delta_p,$$

where $\delta_p$ denotes the Dirac-delta function at $p$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ denotes the multi-indices.

In the following discussion, to avoid some possible confusion with the coordinate $z$, we denote by $z$, instead of $z$, the set of marked points $\{ z_1, \ldots, z_m \}$ which has been used so far.

Consider the moduli space

$$\tilde{M}_{g,m}(J; \beta) = \{( (j, u), z) \mid u : \Sigma \to M, \partial_{(j, J)} u = 0, z = (z_1, \ldots, z_m) \in \text{Conf}_m(\Sigma) \}.$$

The evaluation map $ev_i : \tilde{M}_{g,m}(J; \beta) \to M$ is defined by

$$ev_i((j, u), z) = u(z_i).$$

We then define the universal moduli space

$$\tilde{M}_{g,m}(\beta) = \bigcup_{J \in J_\omega} \tilde{M}_{g,m}(J; \beta) \to J_\omega$$

and define $\tilde{M}_{g,m}^{\text{inj}}(\beta)$ as the open subset of $\tilde{M}_{g,m}(\beta)$ consisting of somewhere-injective curves.

The following is the main theorem of this section.

**Theorem 10.5.2** (Evaluation transversality) The derivatives

$$D(ev_i) : T((j, u), J, \beta) \tilde{M}_{g,m}(\beta) \to T_{u(z_i)} M$$

are surjective for all $i = 1, \ldots, m$ at every $((j, u), J, \beta) \in \mathcal{M}_{g,m}^{\text{inj}}(\beta)$.

Consider the map

$$\Upsilon : J_\omega \times \mathcal{F}_{g,\ell}(\Sigma, M; \beta) \to \mathcal{H}'' \times M^\ell ; (J, (j, u), \beta) \mapsto (\partial_{(j, J)} u, \{ u(z_i) \}).$$
The union of standard moduli spaces $\mathcal{M}_\ell(M, J; \beta)$ over $J \in \mathcal{J}_\omega$ is nothing but
\[
\Upsilon^{-1}(o_{\mathcal{H}''} \times M)/\text{Aut}(\Sigma),
\] (10.5.16)
where $o_{\mathcal{H}''}$ is the zero section of the bundle $\mathcal{H}''$ defined above, and $\text{Aut}(\Sigma)$ acts on $((\Sigma, j), u)$ by conformal reparameterization for any $j$.

First, consider the case of one marked point and denote the corresponding map $\Upsilon$ by $\Upsilon_1$. We have
\[
\tilde{\mathcal{M}}_{g,1}(M; \beta) = \Upsilon^{-1}(o_{\mathcal{H}''} \times M),
\]
\[
\tilde{\mathcal{M}}_{g,1}(M, J; \beta) = \tilde{\mathcal{M}}_1(M; \beta) \cap \pi_2^{-1}(J).
\]

**Proposition 10.5.3** The map $\Upsilon_1$ is transverse to the submanifold $o_{\mathcal{H}''} \times \{p\} \subset \mathcal{H}'' \times M$ for any given point $p \in M$.

**Proof** Its linearization $D\Upsilon_1(J, (j, u), z)$ is given by the map
\[
(B, (b, \xi), v) \mapsto \left(D_{J,(j,u)}\partial(B, (b, \xi)), \xi(u(z)) + du(z)(v)\right)
\] (10.5.17)
for $B \in T_J\mathcal{J}_\omega$, $b \in T_j\mathcal{M}(\Sigma)$, $v \in T_z\Sigma$ and $\xi \in T_u\mathcal{F}(\Sigma, M; \beta)$. This defines a linear map
\[
T_J\mathcal{J}_\omega \times T_j\mathcal{M}(\Sigma) \times T_u\mathcal{F}(\Sigma, M; \beta) \times T_z\Sigma \to \Omega^{(0,1)}(u^*TM) \times T_u\Sigma
\]
on $W^{1,p}$. But, for the map $\Upsilon_1$ to be differentiable, we need to choose the completion of $\mathcal{F}(\Sigma, M; \beta)$ in the $W^{k,p}$-norm for at least $k \geq 2$.

We would like to prove that this linear map is surjective at every element $(u, z_0) \in \tilde{\mathcal{M}}_1(\Sigma, M; \beta)$, i.e., at the pair $(u, z_0)$ satisfying
\[
\partial\tilde{J}(j, u) = 0, \quad u(z_0) = x.
\]
For this purpose, we need to study the solvability of the system of equations
\[
D_{J,(j,u)}\partial(B, (b, \xi)) = \gamma, \quad \xi(u(z_0)) + du(v) = X_0
\] (10.5.18)
for given $\gamma \in \Omega^{(0,1)}(u^*TM)$ and $X_0 \in T_u(z_0)\Sigma$.

For the study of evaluation transversality, the domain complex structure $j$ does not play much of a role in our study. It plays no role throughout our calculations except that it appears as a parameter. Therefore we will fix $j$ throughout the proof. Then it will suffice to consider the case $b = 0 = v$. Then the above equation is reduced to
\[
D_{J,u}\partial(B, \xi) = \gamma, \quad \xi(u(z_0)) = X_0.
\] (10.5.19)
Now we study (10.5.19) for $\xi \in W^{2,p}$. We regard

$$\Omega^{(0,1)}_{1,p}(u^*TM) \times T_{u(z_0)}M =: \mathcal{B}_0$$

as a Banach space with the norm $|| \cdot ||_{1,p} + | \cdot |$, where $| \cdot |$ is any norm induced by an inner product on $T_xM \cong \mathbb{R}^n$. We will show that the image of the map (10.5.17) restricted to the elements of the form $(B, (0, \xi), 0)$ is surjective as a map

$$T_J\mathcal{J}_\omega \times \Omega^0_{2,p}(u^*TM) \rightarrow \Omega^{(0,1)}_{1,p}(u^*TM) \times T_{u(z_0)}M,$$

where $(u, j, z_0, J)$ lies in $\Upsilon^{-1}(o_H'' \times \{p\})$. For clarification of the notation, we denote the natural pairing $\langle \cdot, \cdot \rangle$ and the inner product on $T_xM$ by $(\cdot, \cdot)$.

We will first prove that the image is dense in $\mathcal{B}_0$. Let $(\eta, Y_p) \in \Omega^{(0,1)}_{1,p}(u^*TM) \times T_{p}M$ satisfy

$$\langle D_u \partial_{(j,J)}^\dagger \eta + \frac{1}{2} B \cdot du \circ j, \eta \rangle + \langle \xi, \delta_{z_0} Y_p \rangle = 0 \quad (10.5.20)$$

for all $\xi \in \Omega^0_{2,p}(u^*TM)$ and $B$, where $\delta_{z_0}$ is the Dirac-delta function supported at $z_0$. Without any loss of generality, we may assume that $\xi$ is smooth as before, since $C^\infty(u^*TM) \hookrightarrow \Omega^0_{2,p}(u^*TM)$ is dense. Under this assumption, we would like to show that $\eta = 0 = Y_p$.

Taking $B = 0$ in (10.5.20), we obtain

$$\langle D_u \partial_{(j,J)}^\dagger \xi, \eta \rangle + \langle \xi, \delta_{z_0} Y_p \rangle = 0 \quad \text{for all } \xi \text{ of } C^\infty. \quad (10.5.21)$$

Therefore, by virtue of the definition of the distribution derivatives, $\eta$ satisfies

$$(D_u \partial_{(j,J)})^\dagger \eta - \delta_{z_0} Y_p = 0$$

as a distribution, i.e.,

$$(D_u \partial_{(j,J)})^\dagger \eta = \delta_{z_0} Y_p,$$

where $(D_u \partial_{(j,J)})^\dagger$ is the formal adjoint of $D_u \partial_{(j,J)}$ whose symbol is the same as $D_u\partial_{(j,J)}$ and hence is an elliptic first-order differential operator. By the elliptic regularity, $\eta$ is a classical solution on $\Sigma \setminus \{z_0\}$. We also recall that $\partial^\dagger = -\partial$.

On the other hand, by setting $\xi = 0$ in (10.5.20), we get

$$\langle B \cdot du \circ j, \eta \rangle = 0 \quad (10.5.22)$$

for all $B \in T_{J}\mathcal{J}_\omega$. From this identity, the argument used in the transversality proven in the previous section shows that $\eta = 0$ in a small neighborhood of any
somewhere-injective point in $\Sigma \setminus \{z_i\}$. Such a somewhere-injective point exists by virtue of the hypothesis of $u$ being somewhere injective and the fact that the set of somewhere-injective points is open and dense in the domain under the given hypothesis. Then, by applying the unique continuation theorem, we conclude that $\eta = 0$ on $\Sigma \setminus \{z_0\}$ and hence the support of $\eta$ as a distribution on $\Sigma$ is contained at the one-point subset $\{z_0\}$ of $\Sigma$.

The following lemma will conclude the proof.

**Lemma 10.5.4** $\eta$ is a distributional solution of $(D_u \partial_{(j,J)})^* \eta = 0$ on $\Sigma$ and hence is continuous. In particular, we have $\eta = 0$ in $\big( \Omega^{(0,1)}_{(1,0)}(u^*TM) \big)^*$.

Once we know $\eta = 0$, (10.5.20) is reduced to the finite-dimensional equation

$$\langle (\xi(z_0), Y_p)_{z_0} = 0. \rangle (10.5.23)$$

It remains to show that $Y_p = 0$. For this, we need only show that the image of the evaluation map

$$\xi \mapsto \xi(z_0)$$

is surjective onto $T_p M$, which is now obvious. $\square$

Now it remains to prove Lemma 10.5.4.

**Proof of Lemma 10.5.4** Our primary goal is to prove

$$\langle D_u \partial_{(j,J)} \xi, \eta \rangle = 0 \quad (10.5.24)$$

for all smooth $\xi \in \Omega^0(u^*TM)$, i.e., $\eta$ is a distributional solution of $(D_u \partial_{(j,J)})^* \eta = 0$ on the whole $\Sigma$, not just on $\Sigma \setminus \{z_0\}$.

We start with (10.5.21)

$$\langle D_u \partial_{(j,J)} \xi, \eta \rangle + \langle \xi, \delta_{z_0} Y_p \rangle = 0 \quad \text{for all } \xi \in C^\infty. \quad (10.5.25)$$

We first simplify the expression of the pairing $\langle D_u \partial_{(j,J)} \xi, \eta \rangle$, knowing that supp $\eta \subset \{z_0\}$.

Let $z$ be a complex coordinate centered at a fixed marked point $z_0$ and let $(w_1, \ldots, w_n)$ be the complex coordinates on $M$ regarded as coordinates on a neighborhood of $p$. We consider the standard metric

$$h = \frac{\sqrt{-1}}{2} dz d\bar{z}$$

on a neighborhood $U$ of $z_0$, and with respect to the coordinates $(w_1, \ldots, w_n)$ we fix any Hermitian metric on $\mathbb{C}^n$. 


The following lemma will be crucial in our proof.

**Lemma 10.5.5** Let $\eta$ be as above. For any smooth section $\xi$ of $u^*(TM)$ and $\eta$ of $\left(\Omega^{(0,1)}_{1,p}(u^*TM)\right)^*$

$$\langle D_u \overline{\partial_{(j,J)}} \xi, \eta \rangle = \langle \overline{\partial} \xi, \eta \rangle,$$

where $\overline{\partial}$ is the standard Cauchy–Riemann operator on $\mathbb{C}^n$ in the above coordinate.

**Proof** We have already shown that $\eta$ is a distribution with $\text{supp} \, \eta \subset \{z_0\}$. By the structure theorem on the distribution supported at a point $z_0$, Theorem 10.5.1, we have

$$\eta = P \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) (\delta_{z_0})$$

where $z = s + it$ denotes the given complex coordinates at $z_0$ and $P (\partial/\partial s, \partial/\partial t)$ is a differential operator associated by the polynomial $P$ of two variables with coefficients in $\left(\Lambda^{(0,1)}_{1,1}(u^*TM)\right)^*$.

Furthermore, since $\eta \in (W^{1,p})^*$, the degree of $P$ must be zero and so we obtain

$$\eta = \beta_{z_0} \cdot \delta_{z_0} \tag{10.5.26}$$

for some constant vector $\beta_{z_0} \in \Lambda^{(0,1)}(T_pM)$. This is because the ‘evaluation at a point of the derivative’ of the $W^{1,p}$ map does not define a continuous functional on $W^{1,p}$.

We can write

$$D_u \overline{\partial_{(j,J)}} \xi = \overline{\partial} \xi + E \cdot \partial \xi + F \cdot \xi$$

near $z_0$ in coordinates, where $E$ and $F$ are zeroth-order matrix operators with $E(z_0) = 0 = F(z_0)$. Therefore, by (10.5.26), we derive

$$\langle E \cdot \partial \xi + F \cdot \xi, \eta \rangle = (E(z_0) \partial \xi(z_0) + F(z_0) \xi(z_0), \beta_{z_0}) = 0.$$

Therefore we obtain

$$\langle D_u \overline{\partial_{(j,J)}} \xi, \eta \rangle = \langle \overline{\partial} \xi + E \cdot \partial \xi + F \cdot \xi, \eta \rangle = \langle \overline{\partial} \xi, \eta \rangle,$$

which finishes the proof. □

By this lemma, (10.5.25) becomes

$$\langle \overline{\partial} \xi, \eta \rangle + \langle \xi, \delta_{z_0} Y_p \rangle = 0 \quad \text{for all } \xi. \tag{10.5.27}$$

We decompose $\xi$ as

$$\xi(z) = (\xi(z) - \chi(z) \xi(z_0)) + \chi(z) \xi(z_0)$$

on $U$, where $\chi$ is a cut-off function with $\chi \equiv 1$ in a small neighborhood $V \subset U$ of $z_0$ and satisfies $\text{supp } \chi \subset U$. Then the first summand $\tilde{\xi}$ defined by $\tilde{\xi}(z) := \xi(z) - \chi(z)\xi(z_0)$ is a smooth section on $\Sigma$, and satisfies

$$\tilde{\xi}(z_0) = 0, \quad \overline{\partial} \tilde{\xi} = \overline{\partial} \xi \quad \text{on } V,$$

since $\chi(z)\xi(z_0) \equiv \xi(z_0)$ on $V$. Therefore, by applying (10.5.27) to $\tilde{\xi}$ instead of $\xi$, we obtain

$$\langle \overline{\partial} \tilde{\xi}, \eta \rangle + \langle \tilde{\xi}, \delta_{z_0} Y_p \rangle = 0.$$

Again using the support property $\text{supp } \eta \subset \{z_0\}$ and (10.5.25), we derive

$$\langle \tilde{\xi}, \delta_{z_0} Y_p \rangle = \langle \tilde{\xi}(z_0), Y_p \rangle = 0,$$

and so $\langle \overline{\partial} \tilde{\xi}, \eta \rangle = 0$. But we also have

$$\langle \overline{\partial} \tilde{\xi}, \eta \rangle = \langle \overline{\partial} \xi, \eta \rangle,$$

since $\overline{\partial} \tilde{\xi} = \overline{\partial} \xi$ on $V$ and $\text{supp } \eta \subset \{z_0\}$. Hence we obtain $\langle \overline{\partial} \xi, \eta \rangle = 0$ and thus we have finished the proof of (10.5.24) by Lemma 10.5.5.

By virtue of the elliptic regularity, $\eta$ must be smooth. Since we have already shown $\eta = 0$ on $\Sigma \setminus \{z_0\}$, the continuity of $\eta$ proves that $\eta = 0$ on the whole $\Sigma$.

This lemma finishes the proof of Lemma 10.5.4.

Then Lemma 10.5.4 in turn finishes the proof of Proposition 10.5.3.

Another version of evaluation transversality, which plays an important role in the gluing problem, is the so-called diagonal transversality. We now explain this transversality statement.

Since the details of the proof of this transversality are a variation of Proposition 10.5.3, we will be brief.

We recall from Chapter 9 that each stable map limit is associated with a dual graph that encodes the intersection pattern of the stable map. Each vertex of the dual graph corresponds to an irreducible component and each edge issued at the vertex corresponds to a node of the domain of the stable map, which is a pre-stable curve.

Motivated by this consideration, we consider the $k$-marked moduli space

$$\tilde{\mathcal{M}}_{g,k}(J; \beta).$$

We then consider another $k$ moduli spaces $\tilde{\mathcal{M}}_{g,i+1}(J; \alpha_i)$ with one distinguished marked point for each $i = 1, \ldots, k$. We denote by $\text{ev}_0$ the evaluation
map at the distinguished marked point. The local configuration around the
given vertex of the dual graph is given by the fiber product
\[ \tilde{M}_{g,k}(J;\beta)_{ev} \times_{\prod_{i=1}^{k} ev_{0}^{(i)}} \prod_{i=1}^{k} \tilde{M}_{g,\ell+1}(J;\alpha_{i}). \]  
(10.5.30)

We would like to show that this fiber product bears a manifold structure. For
this purpose, we consider the evaluation maps
\[ Ev : \tilde{M}_{g,k}(J;\beta) \times_{\prod_{i=1}^{k} \tilde{M}_{g,\ell+1}(J;\alpha_{i})} \to (M \times M)^{k} \]
defined by
\[ Ev((u,z);\{(u_{i},z^{(i)})\}) = \prod_{i=1}^{k} \left( ev_{i}(z_{i}), ev_{0}^{(i)}(z_{0}^{(i)}) \right). \]  
(10.5.31)

**Corollary 10.5.6** Suppose that all of \( \tilde{M}_{g,k}(J;\beta), \tilde{M}_{g,\ell+1}(J;\alpha_{i}) \) for \( i = 1, \ldots, k \) are Fredholm-regular. Then the fiber product (10.5.30) is smooth if
\( Ev \) is transversal to the multi-diagonal \( \Delta_{M \times M}^{k} \subset (M \times M)^{k} \).

We now study the transversality of \( Ev \) for a generic choice of \( J \)s. For this
purpose, we consider the map
\[ \Upsilon_{k} : J_{\omega} \times F_{g,k}(M;\beta) \times \prod_{i=1}^{k} F_{g,\ell+1}(M;\alpha_{i}) \]
\[ \to H''_{g}(M;\beta) \times \prod_{i=1}^{k} H''_{g}(M;\alpha_{i}) \times (M \times M)^{k} \]
defined by
\[ \Upsilon_{k}(J,(u,z);\{(u_{i},z^{(i)})\}) = \left( \tilde{\partial}_{(j,J)}(u), \{ \tilde{\partial}_{(j,J)}(u_{i}) \}_{i=1,\ldots,k}; Ev((u,z);\{(u_{i},z^{(i)})\}) \right). \]  
(10.5.32)

Since the proof of Proposition 10.5.3 is local at each \( z_{i} \) in that the perturbation
\( B \) can be localized at each \( z_{i} \) separately, we immediately obtain the following
transversality result.

**Theorem 10.5.7** The map \( \Upsilon_{k} \) is transversal to
\[ o_{H''_{g}(M;\beta)} \times \prod_{i=1}^{k} o_{H''_{g}(M;\alpha_{i})} \times (\Delta_{M})^{k} \]
on $\tilde{M}_{g,k}(M;\beta) \times \prod_{i=1}^{k} \tilde{M}_{g,\ell,1}(M;\alpha_i)$, and hence

$$\Upsilon^{-1}_k \left( \mathcal{O}_{H''_g(M;\beta)} \times \prod_{i=1}^{k} \mathcal{O}_{H''_g(M;\alpha_i)} \times (\Delta_M)^k \right)$$

$$\cap \tilde{M}_{g,k}(M;\beta) \times \prod_{i=1}^{k} \tilde{M}_{g,\ell,1}(M;\alpha_i)$$

is a smooth submanifold of $\tilde{M}_{g,k}(M;\beta) \times \prod_{i=1}^{k} \tilde{M}_{g,\ell,1}(M;\alpha_i)$. Furthermore, the projection map

$$\Upsilon^{-1}_k \left( \mathcal{O}_{H''_g(M;\beta)} \times \prod_{i=1}^{k} \mathcal{O}_{H''_g(M;\alpha_i)} \times (\Delta_M)^k \right)$$

$$\cap \tilde{M}_{g,k}(M;\beta) \times \prod_{i=1}^{k} \tilde{M}_{g,\ell,1}(M;\alpha_i) \to J$$

is a Fredholm map of index

$$\iota(g;\beta) + 2k + \sum_{i=1}^{k} (\iota(g;\alpha_i) + 2\ell_i) - k \dim M. \quad (10.5.33)$$

The following is an immediate corollary of the Sard–Smale theorem.

**Corollary 10.5.8** There exists a Baire set $J^\text{fiber}_\omega$ of $J_\omega$ such that, for any $\beta, \{\beta_i\}$, the intersection of the fiber product (10.5.30) with

$$\tilde{M}_{g,k}(J;\beta) \times \prod_{i=1}^{k} \tilde{M}_{g,\ell,1}(J;\alpha_i)$$

is a smooth manifold of dimension given by (10.5.33) for any $J \in J^\text{fiber}_\omega$.

### 10.6 The problem of negative multiple covers

In this section, we first show that the assumption of somewhere injectivity is essential for the transversality theorem *via perturbation of almost-complex structures* on $M$. We start with the following well-known theorems in complex geometry. Denote by $O(a)$ the sheaf of holomorphic sections of the complex line bundle of degree $a$.

**Theorem 10.6.1** Every holomorphic vector bundle $E$ over $\mathbb{C}P^1$ splits into $E \cong O(a_1) \oplus \cdots \oplus O(a_n)$ for $n = \text{rank}(E)$ and the set of integers $\{a_1, \ldots, a_n\}$ does not depend on the decompositions.
Theorem 10.6.2  Suppose $M$ is a complex manifold. Let $u: \mathbb{C}P^1 \to M$ and $u^*TM \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$, then $u$ is Fredholm-regular if and only if $a_i \geq -1$ for all $1 \leq i \leq n$.

Example 10.6.3  Let $M$ be the complex projective space $\mathbb{C}P^2$ for $p \in M$ and let $\bar{M}$ be the blow up of $M$ at $p$,

$$\bar{M} \xrightarrow{\pi} M.$$ 

Then $\pi^{-1}(p) = E$ is an exceptional sphere. Fix an embedding $u: S^2 \to M$ so that $u(S^2) = E$. Let $NE$ be the normal bundle of $E$ in $\bar{M}$, then $NE$ is isomorphic to the complex line bundle

$$\mathcal{L} := \{(l, v) \in \mathbb{C}P^1 \times \mathbb{C}^2 | v \in l\},$$

i.e., $NE \cong \mathcal{O}(-1)$. By the regularity criterion of $u$ in Theorem 10.6.2, $u$ is a holomorphic sphere that is Fredholm-regular.

On the other hand, consider a map $\phi: \mathbb{C}P^1 \to \mathbb{C}P^1$ given by $\phi(z) = z^2$ on $\mathbb{C}$ and the composition $u \circ \phi$. Then $(u \circ \phi)^*T\bar{M} \cong T\mathbb{C}P^1 \oplus (u \circ \phi)^*NE \cong \mathcal{O}(2) \oplus \mathcal{O}(-2)$. More generally, the multiple cover of $E$ with multiplicity $a$ will have its normal bundle isomorphic to $\mathcal{O}(-a)$ with $a \geq 2$. Therefore this multiple cover is not Fredholm-regular, again by Theorem 10.6.2.

Now we claim that this multiple cover cannot be removed by any small perturbation of the standard complex structure on $\bar{M}$. This is because the simple cover $u: S^2 \to \bar{M}$ parameterizing the exceptional sphere $E$ is transversal with respect to the standard complex structure and so must persist under a small perturbation of the complex structure to an almost-complex structure. Its index is given by $c_1(E) + 2 = 2 - 1 + 2 = 3$. Therefore the multiple-cover map of multiplicity $a$ has an index given by $2-a+2 = 4-a$. For example, if $a > 4$, the index becomes negative and so the corresponding moduli space must be empty, if the moduli space were transversal. This finishes the proof of the claim.

This problem of negative multiple covers prompts one to seek a new approach to the study of transversality allowing more general perturbations than that of almost-complex structures $J$, which depends only on the target manifold $M$ and so cannot destroy any symmetry induced from the domain.
In this chapter, we illustrate the usage of the machinery of pseudoholomorphic curves by providing the proofs of two basic theorems in symplectic topology. Both of them can be proved by a direct analysis of the compactified moduli space of pseudoholomorphic curves combined with a bit of symplectic topological data. The first one is Gromov’s celebrated non-squeezing theorem. In the proof of this theorem, an existence theorem of a certain type of pseudoholomorphic curve is the most essential analytical ingredient. In addition, it also uses the positivity and homological invariance of the symplectic area of closed $J$-holomorphic rational curves. We closely follow Gromov’s original scheme of the proof. In hindsight, the existence result is an immediate consequence of the non-triviality of the one-point (closed) Gromov–Witten invariant on $S^2 \times T^{2(n-1)}$, which is defined by counting the number of elements of the zero-dimensional moduli space of $J$-holomorphic curves. We refer readers to (Gr85), (Mc90) for more non-trivial applications of the finer-structure study of the moduli space itself to some structure theorems of ambient symplectic 4-manifolds.

The second one is the proof of the nondegeneracy of Hofer’s norm on $\text{Ham}(M, \omega)$ for arbitrary tame symplectic manifolds. The proof of this theorem uses the moduli space of solutions of the Cauchy–Riemann equation perturbed by time-dependent Hamiltonian vector fields with a Lagrangian boundary condition, and exploits the automatic displaceability of ‘small’ compact Lagrangian submanifolds in arbitrary symplectic manifolds. We closely follow the author’s simplification (Oh97c) of Chekanov’s proof (Che98). We refer readers to Abouzaid’s article (Ab12) for a remarkable application of the finer-structure study of this compactified moduli space to the construction of an exotic Lagrangian sphere in $T^*S^{4k+1}$. His usage is somewhat reminiscent of Donaldson’s original application of the moduli space of anti-self-dual Yang–Mills equations in his celebrated construction of exotic $\mathbb{R}^4$ (Do86).
The more indirect application of this machinery is accompanied by some homological algebra in general. The most prominent example involves the Floer homology-type invariants constructed out of a family of the moduli spaces which are interrelated to one another. This involves variants of the \( J \)-holomorphic curve equation usually perturbed by Hamiltonian vector fields or by other (non-linear) zeroth-order perturbations of the section \( H_{(j,J)}^{(0,1)} \) and also allowing the domain of the maps to become a punctured Riemann surface. The Floer homology theory is the most celebrated example in which this perturbed Cauchy–Riemann equation is used systematically in the package of homological algebra. In our proof of the nondegeneracy of Hofer’s norm on \( \text{Ham}(M, \omega) \), we illustrate a direct usage of the moduli space of this perturbed Cauchy–Riemann equation by a domain-dependent family of Hamiltonian vector fields without involving homological algebra.

### 11.1 Gromov’s non-squeezing theorem

In this section, closely following Gromov’s original proof (Gr85), we provide a complete proof of Gromov’s celebrated non-squeezing theorem.

**Theorem 11.1.1 (Non-squeezing theorem)** Let \( \omega_0 \) be the standard symplectic form of \( \mathbb{C}^n \cong \mathbb{R}^{2n} \), and let \( B^{2n}(R) \) be a standard closed ball in \( \mathbb{C}^n \) of radius \( R \). Let \( Z^{2n}(r) := D^2(r) \times \mathbb{C}^{n-1} \), for \( D^2(r) \subset \mathbb{C} \). Then there exists a symplectic embedding \( \Phi : (B^{2n}(R), \omega_0) \to (Z^{2n}(r + \epsilon), \omega_0) \) for all \( \epsilon > 0 \) if and only if \( R < r \).

We note that, if \( R < r \), there always exists a standard isometric embedding of \( B^{2n}(R) \) into \( Z^{2n}(r + \epsilon) \) for any \( \epsilon > 0 \), so the ‘if’ part is obvious. Therefore it remains to prove the ‘only if’ part. We will prove this by contradiction.

#### 11.1.1 Outline of the proof

Let \( R > r \) and choose any \( \epsilon > 0 \) such that \( r + \epsilon < R \). Suppose to the contrary that there exists a symplectic embedding \( \phi : B^{2n}(R) \to Z^{2n}(r + \epsilon/2) \). Since we assume that \( R > r + \epsilon \) and

\[
\phi(B^{2n}(R)) \subset \text{Int}(Z^{2n}(r + \epsilon)),
\]

there exists some \( \delta > 0 \) such that \( \phi \) extends to a symplectic embedding, still denoted by \( \phi \),

\[
\phi : B^{2n}(R + \delta) \to Z^{2n}(r + \epsilon).
\]
We push-forward \( J_0 \) from \( B^{2n}(R + \delta) \) by \( \phi \) to the image \( \phi(B^{2n}(R + \delta)) \subset Z^{2n}(r + \epsilon/2) \), and consider an almost-complex structure \( J_1 \) on \( \mathbb{C}^n \) such that

\[
J_1 = \begin{cases} 
\phi_*(J_0) & \text{on } \phi(B^{2n}(R + \delta/2)), \\
J_0 & \text{on } \mathbb{C}^n \setminus D^{2n}(r + \epsilon) \times [-K + 1, K - 1]^{2(n-1)}
\end{cases}
\]

and \( J_1 \) is smoothly extended to the remaining region so that \( J_1 \) is still compatible with \( \omega \).

**Exercise 11.1.2** Prove that this smooth extension is possible. (Hint. Use the polar decomposition and note that the decomposition is canonical.)

In the above, the constant \( K > 0 \) is chosen such that

\[
\phi(B^{2n}(R + \delta)) \subset D^{2n}(r + \epsilon/2) \times [-K + 1, K - 1]^{2(n-1)}. \tag{11.1.2}
\]

Here are the key three steps to prove the non-squeezing theorem.

**Step I (Existence theorem).** Prove that there exists a \( J_1 \)-holomorphic map \( f : (D^2, \partial D^2) \to (\mathbb{C}^n, \mathbb{C}^n \setminus \phi(B^{2n}(R + \delta))) \) such that

\[
\int_C \omega_0 \leq \pi(r + \epsilon)^2, \tag{11.1.3}
\]

where \( C = \text{Image } f \), and

\[
\phi(0) \in C. \tag{11.1.4}
\]

Here we recall that \( \pi(r + \epsilon)^2 \) is the area of the flat disc in \( Z^{2n}(r + \epsilon) \).

**Step II (Monotonicity formula).** Now consider the pre-image of \( C \), i.e.,

\[
\phi^{-1}(C) \cap B^{2n}(R).
\]

Since \( \text{Image}(f|_{\partial D^2}) \subset \mathbb{C}^n \setminus \phi(B^{2n}(R + \delta)) \), \( \phi^{-1}(C) \cap B^{2n}(R) \) defines a proper surface in \( B^{2n}(R) \) passing through the origin. Furthermore, since \( J_1|_{\phi(B^{2n}(R + \delta))} \equiv \phi_* J_0 \) and \( C \) is a \( J_1 \)-holomorphic curve, \( \phi^{-1}(C) \) is a \( J_0 \)-holomorphic curve on \( \mathbb{C}^n \) (and hence a minimal surface with respect to the standard metric on \( \mathbb{C}^n \)). The classical monotonicity formula immediately implies

\[
\text{Area}(\phi^{-1}(C) \cap B^{2n}(R)) \geq \pi R^2
\]

because \( \phi^{-1} \cap B^{2n}(R) \) is a proper \( J_0 \)-holomorphic (and hence minimal) surface passing through the origin. See Figure 11.1.
Step III (Use of $J$-positivity of $\omega_0$). Get a contradiction: we apply (11.1.3), (11.1.4), the fact that $\phi$ is symplectic, change of variables, positivity of $\omega$ on $C$ and (11.1.3), respectively, and derive

$$\pi R^2 \leq \text{Area } \phi^{-1}(C) \cap B^{2n}(R) = \int_{\phi^{-1}(C) \cap B^{2n}(R)} \omega_0 = \int_{\phi^{-1}(C) \cap B^{2n}(R)} \phi^* \omega_0$$

$$= \int_{C \cap \phi(B^{2n}(R))} \omega_0 \leq \int_{C} \omega_0 \leq \pi r^2.$$ 

This gives rise to a contradiction since we assume that $R > r$, and finishes the proof.

11.1.2 Existence scheme and one-point GW invariant

Now, it remains to prove the existence statement in Step I. We first transform the existence statement on the $J$-holomorphic discs into a more tractable existence problem of a $J$-holomorphic sphere on a closed symplectic manifold. By now, we have an almost-complex structure $J_1$ on $Z^{2n}(r) = D^2(r) \times \mathbb{C}^{n-1}$ such that $J_1$ is standard on $\mathbb{C}^n \setminus D^2(r + \epsilon) \times [-K + 1, K - 1]^{2(n-1)} =: D_{r, \epsilon, K}$. Therefore, by the translational invariance of $J_0$ we can push-forward $J_1$ on $D_{r, \epsilon, K}$ to

$$D^2(r + \epsilon) \times T^{2(n-1)}(K),$$

which we still denote by $J_1$, where $T^{2(n-1)}(K)$ is the torus obtained by identifying $-K$ with $K$ in $[-K, K]$, i.e.,
$T^{2(n-1)}(K) = S^1_{K} \times \cdots \times S^1_{K}$

where $S^1_{K} = \mathbb{R}/K\mathbb{Z}$. Note that, since $\phi(B^{2n}(R + \delta)) \subset D_{r,\epsilon,K}$, we may regard $\phi(B^{2n}(R + \delta))$ as a subset of

$D^2(r + \epsilon/2) \times T^{2(n-1)}(K)$.

The corresponding almost-complex structure $J_1$ on this space is still standard near the boundary of the space.

Now, we embed $D^2(r + \epsilon/2)$ into $S^2((r + \epsilon)/2)$ by an area-preserving map and then embed $D^2(r + \epsilon/2) \times T^{2(n-1)}(K)$ into

$P_{r+\epsilon,K} := S^2 \left( \frac{r + \epsilon}{2} \right) \times T^{2(n-1)}(K)$

by the symplectic map $\psi \times id : D^2(r + \epsilon/2) \times T^{2(n-1)}(K) \to P_{r+\epsilon,K}$. We denote by $\omega_{r+\epsilon,K} = \omega_1 \oplus \omega_2$ the product symplectic structure on $P_{r+\epsilon,K}$, where $\omega_1$ and $\omega_2$ are the standard symplectic structure on $S^2((r + \epsilon)/2)$ and $T^{2(n-1)}(K)$, respectively. We now extend the structure $(\psi \times id)_J$ on $(\psi \times id)(D^2(r + \epsilon) \times T^{2(n-1)}(K))$ to the whole $P_{r+\epsilon,K} = S^2((r + \epsilon)/2) \times T^{2(n-1)}(K)$ so that the extension, denoted by $\tilde{J}_1$ is compatible with $\omega_{r+\epsilon,K}$. Now note that $\pi_2(P_{r+\epsilon,K}) = \mathbb{Z}$ and the homotopy class $[S^2((r + \epsilon)/2) \times \{pt]\}$ generates $\pi_2$. Denote this homotopy class by $A$ and let $p_0 \in P_{r+\epsilon,K}$ be the point corresponding to $\phi(0)$ in $Z^{2n}(r + \epsilon)$.

**Assertion.** To finish Step I, it suffices to find a $\tilde{J}_1$-holomorphic sphere $\tilde{C}$ with $[\tilde{C}] = A \in \pi_2(P_{r+\epsilon,K})$ and with $p_0 \in \tilde{C}$.

**Proof of Assertion** Suppose that there exists a $\tilde{J}_1$-holomorphic sphere $\tilde{C}$ and let $u : S^2 \to P_{r+\epsilon,K}$ be the map representing $\tilde{C}$, i.e., $\tilde{C} = \text{Image}(u)$. Then, since $[\tilde{C}] = A$, we have

$$\int_{\tilde{C}} \omega_{r+\epsilon,K} = \pi(r + \epsilon)^2.$$

Since $[\tilde{C}] = A = [S^2((r + \epsilon)/2) \times \{pt]\}$, it is easy to prove that the composition $\pi \circ u : S^2 \to S^2((r + \epsilon)/2)$ is surjective. Now the curve $C$ required in Step I can be chosen to be

$$C = (\psi \times id)^{-1}(\tilde{C}) \subset D^2(r + \epsilon/2) \times T^{2(n-1)}(K),$$

which can be regarded as a subset of $Z^{2n}(r + \epsilon/2)$. Obviously, $C$ is a proper surface in $Z^{2n}(r + \epsilon/2)$ and so $C \cap \phi(B^{2n}(R + \delta))$ defines a proper surface in $\phi(B^{2n}(R + \delta))$ since we assumed $\phi(B^{2n}(R + \delta)) \subset Z^{2n}(r + \epsilon/2)$ at the beginning of this section.
Furthermore, since $\psi \times id$ is a symplectic map and from the definition of $\tilde{J}_1$, we have
\[
\int_{C} \omega_0 = \int_{C} (\psi \times id)^* \omega_{r+\epsilon,K} \\
= \int_{(\psi \times id)^{-1}(C)} (\psi \times id)^* \omega_{r+\epsilon,K} \\
= \int_{C \cap (\psi \times id)(D^2(r+\epsilon) \times T^{2(n-1)}(K))} \omega_{r+\epsilon,K} \\
\leq \int_{\tilde{C}} \omega_{r+\epsilon,K} = \pi (r + \epsilon)^2,
\]
which finishes the proof of (11.1.3). Here the inequality follows from $p_0 \in \tilde{C}$, the monotonicity formula and positivity.

Now we have reduced the proof of the non-squeezing theorem to the following general existence theorem of $J_1$-holomorphic spheres.

**Theorem 11.1.3** Let $P_{r+\epsilon,K} = S^2((r + \epsilon)/2) \times T^{2(n-1)}(K)$ with the symplectic form $\omega_{r+\epsilon,K} = \omega_1 \oplus \omega_2$, let $J_\beta$ be any given compatible almost-complex structure and let $p_0 = (x_0, q_0) \in P_{r+\epsilon,K}$ be a given point in $P_{r+\epsilon,K}$. Let $A = [S^2((r + \epsilon)/2) \times \{pt\}]$ be the generator of $\pi_2(P_{r+\epsilon,K})$. Then there exists a $J_\beta$-holomorphic sphere $u : S^2 \to P_{r+\epsilon,K}$ with $[u] = A$ and $p_0 \in \text{Image}(u)$.

We start with the following general definition.

**Definition 11.1.4** Let $J$ be a compatible almost-complex structure of $(M, \omega)$. A homotopy class $A$ is called $J$-simple if there is no decomposition of $A$ into $A = \sum_{j=1}^{N} A_j$ with $N \geq 2$ such that $A_j$ allows a non-trivial $J$-holomorphic curve. If this holds for any $J \in J_\omega$, then we call the homotopy class $A$ simple.

**Lemma 11.1.5** The homotopy class $A = [S^2 \times \{pt\}]$ is simple.

**Proof** If there could be such a decomposition, then we would have
\[
[\omega](A) = \sum_{j=1}^{N} [\omega](A_j).
\]
But, if $A_j$ allows any non-constant $J$-holomorphic curve, then
\[
[\omega](A_j) > 0 \quad \text{and so} \quad [\omega](A_j) \geq \pi (r + \epsilon)^2.
\]
On the other hand, we have $[\omega](A) = \pi (r + \epsilon)^2$ and hence such a decomposition is not possible.
One immediate corollary of this simpleness of \( A \) is the following.

**Corollary 11.1.6** The set 
\[
\widetilde{\mathcal{M}}(M; A) := \{(u, J) \mid J \in \mathcal{J}_\omega, \partial_J u = 0, [u] = A\}
\]
coincides with its open subset \( \widetilde{\mathcal{M}}^{\text{inj}}(M; A) \). In particular it is a smooth infinite-dimensional (Banach) manifold and the projection \( \Pi_A : \widetilde{\mathcal{M}}(M; A) \to \mathcal{J}_\omega \) is a smooth Fredholm map of index \( 2c_1(A) + 2n = 4 + 2n \).

Now, we go back to the proof of Theorem 11.1.3. To prove this, we will use a version of the well-known *continuity method* in PDEs. We start with the standard (integrable) product structure \( J_\alpha \) on \( S^2((r + \epsilon)/2) \times T^2(n-1)(K) \).

**Lemma 11.1.7** The product structure \( J_\alpha \) on \( S^2((r + \epsilon)/2) \times T^2(n-1)(K) \) is \( A \)-regular, where \( A = [S^2((r + \epsilon)/2) \times \{pt\}] \), and \( \widetilde{\mathcal{M}}(J_\alpha; A) \) becomes 
\[
\widetilde{\mathcal{M}}(J_\alpha; A) = \left\{ f : S^2 \to P_{r+\epsilon,K} \mid f(z) = (f_1(z), q), \quad q \in T^{2(n-1)}(K), \quad f_1 : S^2 \to S^2 \left( \frac{r + \epsilon}{2} \right) \text{ is a biholomorphism} \right\}.
\]

**Exercise 11.1.8** Prove this lemma.

Note that the index formula \( 2(2 + n) = 2n + 4 \) is consistent with the explicit description of \( \widetilde{\mathcal{M}}(J_\alpha; A) \) in the above lemma. Recall that we are interested in finding a \( J_\beta \)-holomorphic sphere passing through a given point \( p_0 \). So we consider the following evaluation map:
\[
ev_A : \mathcal{M}_1(J_\alpha; A) \to P, \quad \ev_A(f, z) = f(z),
\]
where \( \mathcal{M}_1(J_\alpha; A) = \widetilde{\mathcal{M}}(J; A) \times S^2/PSL(2; \mathbb{C}) \) is the moduli space of one-point marked stable maps in class \( A \). The dimension of \( \mathcal{M}_1(J_\alpha; A) \) is given by 
\[
2(c_1(A) + n) + 2 - 6 = 2c_1(A) + 2n - 4.
\]
For the product structure \( J_\alpha \) in the circumstance of Lemma 11.1.7, where \( c_1(A) = 2 \), one can explicitly see that this dimension becomes \( 2n \). In fact, we can prove the following lemma.

**Lemma 11.1.9** For the product structure \( J_\alpha \), \( \mathcal{M}_1(J_\alpha; A) \) is a compact 2\( n \)-manifold and the evaluation map 
\[
ev_A : \mathcal{M}_1(J_\alpha; A) \to P_{r+\epsilon,K}
\]
becomes a diffeomorphism. In particular, it has non-zero \( \mathbb{Z}_2 \)-degree.
Exercise 11.1.10  Prove this lemma.

Now, we consider the given $J_\beta$ in Theorem 11.1.3. In general, $J_\beta$ need not be $A$-regular and so we choose a sequence of $A$-regular $J_i$ so that

$$J_i \to J_\beta \quad \text{as} \quad i \to \infty$$

in $C^\infty$ topology. Now, for each fixed $i$, we consider a path $\overline{J} = \{J_i\}_{0 \leq t \leq 1}$ such that $J_0 = J_\alpha$ and $J_1 = J_i$ and consider the diagram

$$\begin{array}{ccc}
\mathcal{M}_1(M; A) & \xrightarrow{\Pi_A} & F_w \\
\downarrow & & \\
[0, 1] & \xrightarrow{\overline{J}} & \\
\end{array}$$

where $\mathcal{M}_1(M; A) = \bigcup_{J \in \mathcal{J}_0} J \times \mathcal{M}_1(J; A)$.

By the relative transversality theorem, we can choose the path $\overline{J}$ that is transverse to $\Pi_A$ and $\overline{J}_0 = J_\alpha$, $\overline{J}_1 = J_i$. Then the family $\Pi^{-1}_A(\overline{J}) = \bigcup_{t \in [0, 1]} \{t\} \times \mathcal{M}_1(J_t; A)$ defines a smooth cobordism between $\mathcal{M}_1(J_\alpha; A)$ and $\mathcal{M}_1(J_i; A)$.

Furthermore, the evaluation map

$$\ev^0_1 \sqcup \ev^1_1 : \mathcal{M}_1(J_\alpha; A) \sqcup \mathcal{M}_1(J_i; A) \to P_{r+\epsilon,K}$$

extends to the parameterized evaluation map

$$\Ev_A : \mathcal{M}_1(\overline{J}; A) \times_G S^2 \to P_{r+\epsilon,K} \times [0, 1]; \quad \mathcal{M}_1(\overline{J}; A) := \bigcup_{t \in [0, 1]} \{t\} \times \mathcal{M}_1(J_t; A),$$

defined by

$$\Ev_A(t; f_i, J_i, z) = (f_i(z), t),$$

which is also smooth.

Under the circumstance above, if one can prove that $\mathcal{M}_1(\overline{J}; A)$ is compact, it will provide a compact cobordism between $\mathcal{M}_1(J_\alpha; A)$ and $\mathcal{M}_1(J_i; A)$ and so the evaluation maps

$$\ev^0_A : \mathcal{M}_1(J_\alpha; A) \to P_{r+\epsilon,K} \times \{0\}$$

and

$$\ev^1_A : \mathcal{M}_1(J_i; A) \to P_{r+\epsilon,K} \times \{1\}$$

must have the same $\mathbb{Z}_2$-degree. Therefore since $\ev^0_A$ has $\mathbb{Z}_2$-degree 1 by Lemma 5.6,
must have non-zero $\mathbb{Z}_2$-degree and in particular is surjective. This in turn implies that there exists a $J_i$-holomorphic curve passing through $p_0$. Let us denote such a $J_i$-holomorphic curve by $u_i$. Now, if one can prove the uniform estimate for the derivative $du_i$

$$\max_{z \in S^2} |du_i(z)| \leq C \quad (11.1.5)$$

for all $i$, then, after reparameterization, one can pass to the limit to find a $J_\beta$-holomorphic sphere $u_\infty$ such that $p_0 \in \text{Image } u_\beta$, which will finish the proof of Theorem 11.1.3 and hence the non-squeezing theorem.

Both of the above two compactness statements will be a consequence of Gromov’s compactness theorem, which we will describe in the next section. To apply Gromov’s compactness theorem to our situation, the uniform area bound $(11.1.5)$ and the fact that the homotopy class $A = [S^2((r + \epsilon)/2) \times \{pt\}]$ is simple will be used.

### 11.1.3 Proof of smooth convergence

The main goal of this section is to prove the two compactness statements left out in the last section. We first consider the following proposition.

**Proposition 11.1.11** Let $P = S^2((r + \epsilon)/2) \times T^{2(n-1)}(K)$ and $A \in \pi_2(P)$ be as before. Then for any path $J = \{J_t\}_{0 \leq t \leq 1} \subset J_\omega$, $M_1(J; A)$ is compact. Furthermore, if we fix $J_0 = J_\alpha$, $J_1 = J_\gamma$ that are $A$-regular, there exists a smooth path $\overline{J}$ such that $M_1(\overline{J}; A)$ becomes a compact smooth manifold.

This is a parameterized version of Gromov’s compactness theorem.

**Theorem 11.1.12** (Gromov’s convergence theorem) Let $J_\gamma \rightarrow J_\infty$ be a convergent sequence in $J_\omega$. Then, for any sequence of $J_\gamma$-holomorphic maps $f_\gamma : S^2 \rightarrow P$ with the uniform area bound

$$\text{Area}(f_\gamma) < C,$$

there exists a subsequence of $f_\gamma$ such that the unparameterized curve $C_\gamma = \text{Image } f_\gamma$ weakly converges to a cusp curve $C_\infty = \{C_{\infty,l}\}_{1 \leq l \leq N}$ so that

1. $\text{Area}(C_\infty) = \lim_{\gamma \rightarrow \infty} \text{Area}(C_\gamma) = \sum_{l=1}^{N} \text{Area}(C_{\infty,l})$
2. $[C_\infty] = \sum_{l=1}^{N} [C_{\infty,l}]$ in $\pi_2(P)$
3. $C_\infty$ is a connected cusp curve.
Furthermore, if $N = 1$, i.e., $C_\infty$ is a genuine smooth curve, then the subsequence $f_\gamma C^\infty$-converges to a limit $f_\infty$ after reparameterization if necessary.

**Exercise 11.1.13**  This is an immediate consequence of the compactness theorem of the set $\overline{\mathcal{M}}_1(M, J; \leq C)$ of stable maps of genus zero. Verify this claim.

Now, we are ready to prove Proposition 11.1.11.

**Proof of Proposition 11.1.11**  Since $\mathcal{M}_1(\overline{J}; A)$ is an $S^2$-bundle over $\mathcal{M}(\overline{J}; A) = \overline{\mathcal{M}}(\overline{J}; A)/G$, it suffices to prove that $\mathcal{M}(\overline{J}; A)$ is compact. Let $(f_i, J_i)$ be a sequence in $\mathcal{M}(\overline{J}; A)$. We have the uniform area bound $\omega(f_i) \leq C(A)$. Then the above compactness theorem implies that there exists a subsequence of $f_i$ still denoted by $f_i$ such that the unparameterized curve $C_i = \text{Image } f_i$ converges to a cusp-curve $C_\infty = \sum_{\ell=1}^N C_{\infty, \ell}$. By choosing a subsequence, we may assume that $t_i \to t_\infty$. However, since $A$ is simple, $N$ cannot be bigger than 1 by the argument at the end of the previous section and so the subsequence $f_i$ converges to a $J_\infty$-holomorphic curve after reparameterization, which finishes the proof of the fact that $\mathcal{M}(\overline{J}; A)$ is compact. The last statement will follow by the transversality theorem if we choose a path $\overline{J}$ that is transverse to the projection $\Pi_2 : \mathcal{M}(M; A) \to \mathcal{J}_\omega$ in Proposition 5.2.

We are given a sequence of $A$-regular $J_i$ such that $J_i \to J_\beta$, where $J_\beta$ is as in Theorem 11.1.3.

By applying Proposition 6.1 and the existence scheme for each $J_i$, we obtain a sequence $f_i$ of $J_i$-holomorphic spheres passing through the point $p_0$. Again using the area bound and the simpleness of $A$, Gromov’s compactness theorem as given above guarantees the existence of the limit $f_\infty$ (after reparameterization) such that $f_\infty$ is $J_\beta$-holomorphic and $p_0 \in \text{Image } f_\infty$. This finally finishes the proof of the non-squeezing theorem.

### 11.2 Nondegeneracy of the Hofer norm

We recall the definition of the Hofer norm $\|\phi\|$ of Hamiltonian diffeomorphism $\phi$ from Section 5.4. According to the remark at the end of Section 5.4, the nondegeneracy of the norm on a symplectic manifold can be proved by establishing the positivity of the displacement energy of any closed Lagrangian submanifold therein.

Recall the definition of displacement energy from Definition 5.4.1 specializing to the set of compact Lagrangian submanifolds.
Definition 11.2.1  The displacement energy of a given compact Lagrangian submanifold $L \subset (M, \omega)$ is defined by

$$e(L; M, \omega) = \inf_{H} \{ \|H\| \mid L \cap \phi_{H}^1(L) = \emptyset \}. $$

In (Po93), Polterovich proved that

$$e(L; M, \omega) \geq \frac{1}{2} \Gamma_{(L; M, \omega)} > 0, \quad (11.2.6)$$

where $\Gamma_{(L; M, \omega)}$ is the (positive) generator of the subgroup

$$\{ \omega(\beta) \mid \beta \in \pi_2(M, L) \} \subset \mathbb{R}$$

for any rational Lagrangian submanifold. It is an easy consequence from this to obtain the nondegeneracy of Hofer’s norm for tame rational symplectic manifolds. Via a version of the Floer homology theory, Chekanov improved this result in (Che94) by first removing the factor of $\frac{1}{2}$ in (11.2.6) and then extending his result to arbitrary tame non-rational symplectic manifolds and also giving an estimate of the number of intersections $L \cap \phi_{H}^1(L)$ in relation to Arnol’d’s conjecture.

In this section, we first introduce a variant of pseudoholomorphic curves whose conformal symmetry is broken. Such a variation was first used by Gromov himself in his proof of the non-exactness of any closed Lagrangian embeddings in $\mathbb{C}^n$ (Gr85). Gromov considered a one-parameter family of inhomogeneous equations $\tilde{\partial}u = \lambda g$ on $\mathbb{C}^n$ with a constant vector $g \in \mathbb{C}^n$ and $0 \leq \lambda < \infty$, and compared the corresponding moduli space for $\lambda = 0$ and those for $\lambda \to \infty$. In his proof given in (Gr85), Gromov exploited the fact that this perturbed moduli space becomes empty for all sufficiently large $\lambda$, which he himself also proved by an ingenious application of the Cauchy-integral formula.

Soon Floer provided a more natural variation of the equation perturbed by a Hamiltonian vector field which is used to construct the Floer homology for Hamiltonian fixed points (Fl89b). In this section, we use the cut-off version of Floer’s perturbed Cauchy–Riemann equation, closely following the idea from (Oh97c) for tame symplectic manifolds in the following way.

(1) We first identify emptiness of intersections of two Lagrangian submanifolds $L$ and $\phi_{H}^1(L)$ as the obstruction to compactness of certain parameterized moduli space of perturbed Cauchy–Riemann equations.

(2) We then combine some simple but fundamental calculations to relate the displacement energy, which is dynamical in nature, to the geometric invariant $A(L; M, \omega)$ of $L$. 

This kind of calculation first appeared in \((H85)\) and \((Oh93a)\) in a crude form, but it was Chekanov in \((Che94)\) who did this calculation in its optimal form, which has been used systematically by the present author in a series of papers on spectral invariants and their applications \((Oh97b, Oh05d)\). By now this calculation has become established as one of the fundamental ingredients in applications of Floer homology techniques to various problems in Hamiltonian dynamics and symplectic topology.

This section will also provide a glimpse of what kind of information the Floer homology encodes on the underlying symplectic manifolds, which will be studied in detail in the next couple of parts.

### 11.2.1 Hamiltonian-perturbed pseudoholomorphic curves

Let \(\phi\) be a Hamiltonian diffeomorphism of \((M, \omega)\). Denote by \(H\) a compactly supported time-dependent Hamiltonian function \(H\) with \(H \mapsto \phi\). Let \(L\) be a compact Lagrangian submanifold. We have a one-to-one correspondence between \(L \cap \phi(L)\) and the set of solutions \(z : [0, 1] \to M\) of

\[
\dot{z} = X_H(t, z), \quad z(0), z(1) \in L. \tag{11.2.7}
\]

Here is the precise correspondence:

\[
p \in L \cap \phi(L) \Leftrightarrow z = z^H_p \text{ with } z^H_p(t) := \phi_H((\phi^1_H)^{-1}(p)). \tag{11.2.8}
\]

We will see that the \(L^2\)-gradient equation of the action functional \(\mathcal{A}_H\) is given by the map \(u : \mathbb{R} \times [0, 1] \to M\) satisfying

\[
\begin{align*}
\frac{\partial u}{\partial \tau} + J_t(\frac{\partial u}{\partial t} - X_H(u)) &= 0, \\
u(\tau, 0) &\in L, \quad u(\tau, 1) \in L. \tag{11.2.9}
\end{align*}
\]

In this section, we will not emphasize this origin of the equation but regard this equation as a perturbed Cauchy–Riemann equation of the form

\[
u \mapsto \overline{\partial} \Psi(u) + P^{0,1}(u) = (du + P(u))^{0,1},
\]

where \(P\) is a section of the bundle

\[
\mathcal{H} \to \mathcal{F}; \quad \mathcal{H}_u = \Omega^1(u^*TM)
\]

and \((\cdot)^{(0,1)}\) is the \((0, 1)\)-part of \((\cdot) \in \Omega^1(u^*TM)\). In terms of the coordinate \(z = (\tau, t)\), we can write

\[
P(z, u) = X(z, u)d\tau + Y(z, u)dt,
\]
where $X(z, u), Y(z, u) \in \Omega^0(u^* M)$. Therefore we have

$$2(du + P(u))^{0.1} \left( \frac{\partial}{\partial \tau} \right) = \frac{\partial u}{\partial \tau} + J_t \frac{\partial u}{\partial t} + (X(z, u) + J_t Y(z, u)).$$

Equation (11.2.9) corresponds to the case, e.g., where $P = P_H$ is given by

$$P_H(z, u) = -X_H(t, u) dt.$$ 

We note that $P_H(0, 1) = \frac{1}{2} \int_1^0 |\dot{z} - X_H(z)|^2 dt$.

The following lemma illustrates how the analytical study of the perturbed Cauchy–Riemann equation considered above gives rise to an existence result of Lagrangian intersections.

**Lemma 11.2.2** Suppose that (11.2.9) carries a solution $u : \mathbb{R} \times [0, 1] \to M$ with finite energy

$$E_J(u) = \int_{-\infty}^{\infty} \int_0^1 \left| \frac{\partial u}{\partial \tau} \right|_{J_t}^2 \, dt \, d\tau < \infty. \quad (11.2.10)$$

Then there exists a sequence $\tau_n \to \infty$ (or $\tau_n \to -\infty$) such that the path $z_n := u(\tau_n, \cdot)$ converges in $C^\infty$ topology to a Hamiltonian path $z : ([0, 1], [0, 1]) \to (M, L)$ satisfying (11.2.7). In particular we have $L \cap \phi(L) \neq \emptyset$.

**Proof** Because of the finiteness (11.2.10) and the fact that the integrand is non-negative, there exists a sequence $\tau_n \nearrow \infty$ such that

$$\int_0^1 \left| \frac{\partial u}{\partial \tau} (\tau_n, \cdot) \right|_{J_t}^2 \, dt \to 0.$$

Since $u$ satisfies (11.2.9), this is equivalent to

$$\int_0^1 \left| \frac{\partial u}{\partial t} - X_H(u) \right|_{J_t}^2 \, dt \to 0. \quad (11.2.11)$$

Setting $z_n = u(\tau_n, \cdot)$, we have

$$\int_0^1 |\dot{z}_n - X_H(z_n)|_{J_t}^2 \, dt \to 0. \quad (11.2.12)$$

Since $H$ is compactly supported, we have $\max_{(t,x)} |X_H(t, x)|_{J_t} < C$ for some $C > 0$. Therefore

$$\frac{1}{2} \int_0^1 |\dot{z}_n|_{J_t}^2 \, dt < C^2 + 1 < \infty \quad (11.2.13)$$

for all sufficiently large $n$. By the compactness of the Sobolev embedding

$$W^{1,2}([0, 1], M) \hookrightarrow C^0([0, 1], M)$$

we can take a subsequence, again denoted by $z_n$, so that $z_n \to z$ in $C^0$. 


We now claim that $z$ is differentiable and satisfies $\dot{z} = X_H(z)$. This is a local problem. By taking coordinates, we may assume that $M = \mathbb{R}^{2n}$. To prove the differentiability of $z$, we write

$$z_n(t + h) - z_n(t) - X_H(t, z_n(t))h = \int_0^1 \left( \frac{d}{du} z_n(t + uh) - X_H(t, z_n(t)) \right) h \, du$$

$$= \int_0^1 (hX_H(t + uh, z_n(t + uh)) - hX_H(t, z_n(t))) du$$

$$= h \int_0^1 (X_H(t + uh, z_n(t + uh)) - X_H(t, z_n(t))) du.$$

By the continuity of $X_H$ and the uniform convergence of $z_n \to z$ we have convergence

$$X_H(t + uh, z_n(t + uh)) \to X_H(t + uh, z(t + uh)), \quad X_H(t, z_n(t)) \to X_H(t, z(t))$$

uniformly over $t, u$ as $n \to \infty$. This proves that

$$z(t + h) - z(t) - X_H(t, z(t))h = h \int_0^1 (X_H(t + uh, z(t + uh)) - X_H(t, z(t))) du.$$

Furthermore, by virtue of the continuity, we have

$$\left| \int_0^1 (X_H(t + uh, z(t + uh)) - X_H(t, z(t))) du \right| \to 0$$

as $|h| \to 0$. Therefore we have derived

$$|z(t + h) - z(t) - X_H(t, z(t))h| = o(|h|),$$

which proves that $z$ is differentiable at $t$ and its derivative is given by $\dot{z}(t) = X_H(t, z(t))$. By a similar boot-strapping argument, we can show that $z$ is a smooth solution to Hamilton’s equation for $H$.

Finally, the convergence of $z_n \to z$ in $C^\infty$ topology follows again by the boot-strap method on differentiating the equation $\dot{z}_n = X_H(t, z_n)$, which finishes the proof of the convergence statement. \qed

The following rephrased form of the last statement will play an important role as an obstruction to compactness of the moduli space of solutions of a suitably cut-off version of Floer’s trajectory equation.

**Corollary 11.2.3** Suppose that $L \cap \phi^t_H(L)$ is empty. Then (11.2.9) has no solution of finite energy.
We first consider the special case where $J \equiv J_0$ and $H \equiv 0$. In this case, (11.2.9) becomes

\[
\begin{aligned}
\partial u/\partial \tau + J_0 \partial u/\partial t &= 0, \\
u(\tau, 0) \in L, u(\tau, 1) \in L.
\end{aligned}
\] (11.2.14)

We assume that

\[
\int \int |\partial u/\partial \tau|_{J_0}^2 < \infty.
\]

By composing a conformal diffeomorphism $\phi : D^2 \setminus \{1, -1\} \to \mathbb{R} \times [0, 1]$, we obtain a map $\tilde{u} := u \circ \phi$,

\[
\tilde{u} : (D^2 \setminus \{1, -1\}, \partial D^2 \setminus \{1, -1\}) \to (M, L),
\]

which is $J_0$-holomorphic with $\int |d\tilde{u}|_{J_0}^2 < \infty$. By the removable singularity theorem, $\tilde{u}$ smoothly extends to a $J$-holomorphic disc

\[
\tilde{u} : (D^2, \partial D^2) \to (M, L).
\]

In particular, we can associate a natural homotopy class $[u] = [\tilde{u}] \in \pi_2(M, L)$.

In fact the above discussion applies equally whenever the pair $(J, H)$ satisfies $J \equiv J_0$ and $H \equiv 0$ near $\tau = \pm \infty$. This leads us to consider the non-autonomous version, i.e., the $\tau$-dependent version of Equation (11.2.9). Consider the two-parameter family of almost-complex structures and Hamiltonian functions

\[
J = \{J(s, t)\}, \quad H = sH_t \text{ for } (s, t) \in [0, 1]^2.
\]

Note that $[0, 1]^2$ is a compact set and so $J, H$ are compact families. We will be particularly interested in the case

\[
J(s, t, x) = J_0, \quad \text{in a neighborhood of } \partial[0, 1]^2.
\]

We may also assume $J(s, t)$ is as close to $J_0$ as we want in $C^\infty$ topology. We may also assume that $H_t \equiv 0$ near $t = 0, 1$ by reparameterizing the given Hamiltonian flow $\phi^t_H$ so that it becomes constant near $t = 0, 1$. For each $K \in \mathbb{R}_+ = [0, \infty)$, we define a family of cut-off functions $\rho_K : \mathbb{R} \to [0, 1]$ so that, for $K \geq 1$, they satisfy

\[
\rho_K = \begin{cases} 
0, & \text{for } |\tau| \geq K + 1, \\
1, & \text{for } |\tau| \leq K.
\end{cases}
\] (11.2.15)

We also require

\[
\rho'_K \geq 0 \quad \text{on } [-K - 1, -K], \\
\rho'_K \leq 0 \quad \text{on } [K, K + 1].
\] (11.2.16)

For $0 \leq K \leq 1$, define $\rho_K = K \cdot \rho_1$. Note that $\rho_0 \equiv 0$. 

Then we consider the following one-parameter $K$-family of equations:

\[
\begin{cases}
\frac{\partial u}{\partial \tau} + J_{\phi_k(\tau, t)}(\frac{\partial u}{\partial t} - \rho_K(\tau)X_H(u)) = 0, \\
u(\tau, 0) \in L, \; u(\tau, 1) \in L,
\end{cases}
\]

with the finite energy \( \int |d\tilde{u}| J_{\phi_k(\tau, t)}^2 < \infty \). Note that, if \(|\tau| \geq K + 1\), the equation becomes

\[
\frac{\partial u}{\partial \tau} + J_0 \frac{\partial u}{\partial t} = 0.
\]

In particular, the finite-energy condition allows us to extend \( u \circ \phi \) smoothly to the whole disc. Hence it defines a homotopy class \([u] \in \pi_2(M, L)\).

**Definition 11.2.4** For \( K > 0 \), we define

\[
\mathcal{M}_K(J, H; A) = \left\{ u : \mathbb{R} \times [0, 1] \to M \mid u \text{ satisfies } (11.2.17) \right. \\
\left. \quad \text{and } \int |\frac{\partial u}{\partial \tau}|^2 J < \infty, \; [u] = A \right\}.
\]

The following is the basic structure theorem of \( \mathcal{M}_K(J, H; A) \) whose proof is a slight variation of the generic transversality theorem and hence is left to the reader.

**Theorem 11.2.5**

1. For each fixed \( K > 0 \), there exists a generic choice of \((J, H)\) such that \( \mathcal{M}_K(J, H; A) \) becomes a smooth manifold of dim \( n + \mu_L(A) \) if nonempty. In particular, if \( A = 0 \), \( \dim \mathcal{M}_K(J, H; A) = n \) if nonempty.
2. For the case \( A = 0, K = 0 \), all solutions are constant and Fredholm regular and hence \( \mathcal{M}_K(J, H; A) \cong L \). Furthermore, the evaluation map

\[
\ev : \mathcal{M}_0(J, H; 0) \to L : u \mapsto u(0, 0)
\]

is a diffeomorphism.
3. Let \( K_0 \gg 0 \) and assume \( \mathcal{M}_{K_0}(J, H; A) \) is regular and \( \ev : \mathcal{M}_{K_0}(J, H; A) \to L \) is smooth. Then the parameterized moduli space

\[
\mathcal{M}^\text{para}_{[0, K_0]}(J, H; A) := \bigcup_{K \in [0, K_0]} \{K\} \times \mathcal{M}_K(J, H; A) \to [0, K_0]
\]

is a smooth manifold with boundary

\[
\{0\} \times \mathcal{M}_0(J, H; A) \sqcup \{K_0\} \times \mathcal{M}_{K_0}(J, H; A)
\]
and the evaluation map

\[ \text{Ev} : \mathcal{M}_{[0,K_0]}^{\text{para}}(J,H;A) \times \mathbb{R} \to L \times \mathbb{R}_+ \times \mathbb{R} : ((K,u),\tau) \mapsto (K,u(\tau),\tau) \]

is smooth.

The following a-priori energy bound is a key ingredient in relation to the lower bound of the displacement energy.

**Lemma 11.2.6** Let \( u \) be any finite-energy solution of (11.2.17) with \([u] = A \in \pi_2(M,L)\) fixed. Then we have

\[
E_{J,K}(u) := \int_{-\infty}^{\infty} \int_{0}^{1} \left( \frac{\partial u}{\partial \tau} \right)_J^2 \, dt \, d\tau = \omega(A) - \int_{-\infty}^{\infty} \rho_K(\tau) \int_{0}^{1} (H_t \circ u) \, dt \, d\tau \]

(11.2.18)

In particular, we have

\[
E_{J,K}(u) \leq \omega(A) + ||H||, \tag{11.2.19}
\]

where \( ||H|| := \int_{0}^{1} (\max H_t - \min H_t) \, dt \). When \( A = 0 \), we have \( E_{J,K}(u) \leq ||H|| \). Also the upper bound does not depend on \( K, \rho_K \) or \( u \) as long as \( u \) has finite energy.

**Proof** The proof will be carried out by an explicit calculation. This calculation is the key calculation that relates the energy and the Hofer norm, which we extract from (Oh97c). We compute

\[
\int_{-\infty}^{\infty} \int_{0}^{1} \left( \frac{\partial u}{\partial \tau} \right)_J^2 \, dt \, d\tau = \int_{-\infty}^{\infty} \int_{0}^{1} \omega \left( \frac{\partial u}{\partial \tau}, J_K \frac{\partial u}{\partial \tau} \right) \, dt \, d\tau
\]

\[
= \int_{-\infty}^{\infty} \int_{0}^{1} \omega \left( \frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial \tau} - \rho_K(\tau) X_H(u) \right) \, dt \, d\tau
\]

\[
= \int_{-\infty}^{\infty} \int_{0}^{1} \omega \left( \frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial \tau} \right) \, dt \, d\tau
\]

\[
- \int_{-\infty}^{\infty} \rho_K(\tau) \int_{0}^{1} \omega \left( \frac{\partial u}{\partial \tau}, X_H(u) \right) \, dt \, d\tau
\]

\[
= \omega(A) - \int_{-\infty}^{\infty} \rho_K(\tau) \int_{0}^{1} \left( -dH_t(u) \frac{\partial u}{\partial \tau} \right) \, dt \, d\tau
\]

\[
= \omega(A) + \int_{-\infty}^{\infty} \rho_K(\tau) \int_{0}^{1} \frac{\partial}{\partial \tau} (H_t \circ u) \, dt \, d\tau
\]
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\[ = \omega(A) - \int_{-\infty}^{\infty} \rho'_K(\tau) \int_0^1 (H_t \circ u) dt d\tau \]

\[ = \omega(A) - \int_{-K-1}^{-K} \rho'_K(\tau) \int_0^1 (H_t \circ u) dt d\tau \]

\[ - \int_K^{K+1} \rho'_K(\tau) \int_0^1 (H_t \circ u) dt d\tau. \]

This finishes the proof of (11.2.18). The inequality (11.2.19) immediately follows from this since we have \( \int_{-K-1}^{-K} \rho'_K(\tau) dt = 1 \) and \( \int_K^{K+1} \rho'_K(\tau) dt = -1. \)

**Proposition 11.2.7** Let \( H_t \) be the Hamiltonian such that \( L \cap \phi_t^1(L) \) is empty and \( H = sH_t \) and \( J \) as before. Fix \( A \in \pi_2(M, L) \). Then there exists \( K_0 > 0 \) sufficiently large that \( M_K(J, H; A) \) is empty for all \( K \geq K_0 \).

**Proof** We will use an a-priori bound to apply the following version of the compactness theorem, whose proof is a variation of Gromov’s convergence theorem and will be postponed until Part 3.

**Theorem 11.2.8** (Gromov–Floer compactness theorem) Suppose \( K_\alpha \to K_\infty \in \mathbb{R}_+ \cup \{\infty\} \) and let \( u_\alpha \) be solutions of (11.2.17) for \( K = K_\alpha \) with uniform bound

\[ E_{J_{K_\alpha}}(u_\alpha) < C < \infty \] for \( C \) independent of \( \alpha \).

Then, there exist a subsequence again enumerated by \( u_\alpha \) and a cusp-trajectory \((u, v, w)\) such that

1. \( u \) is a solution of (11.2.17) with \( K = K_\infty \);
2. \( v = \{v_i\}_{i=1}^k \), where each \( v_i \) is a \( J_{(s_i,t_i)} \)-holomorphic sphere and each \( w_j \) is a \( J_0 \)-holomorphic disc with its boundary lying on \( L \);
3. \( \lim_{\alpha \to \infty} E_{J_{K_\alpha}}(u_\alpha) = E_{J_{K_\infty}}(u) + \sum_i \omega(v_i) + \sum_j \omega(w_j) \), where
   \[ \omega(v_i) = \int v_i^* \omega = \frac{1}{2} \int |d\gamma_i|^2 = E_{J_{(s_i,t_i)}}(v_i); \]
4. \( u_\alpha \) converges to \((u, v, w)\) in Hausdorff topology and converges in compact \( C^\infty \) topology away from the nodes.

Furthermore, if \( v = w = \phi \), then \( u_\alpha \to u \) smoothly on \( \mathbb{R} \times [0, 1] \).

Now we are ready to wrap up the proof of Proposition 11.2.7. The proof will be given by contradiction. Suppose that there exists a sequence \( K_\alpha \uparrow \infty \) and a solution \( u_\alpha \in M_{K_\alpha}(J, H; A) \). By the a-priori energy bound for \( M_{K_\alpha}(J, H; A) \), we can apply the Gromov–Floer compactness theorem. In other words, there exists \((u, v, w)\) such that \( u_\alpha \to (u, v, w) \) in the sense of the compactness theorem.
In particular, we have produced a solution $u$ of (11.2.17). But (11.2.17) is nothing but
\begin{align*}
\begin{cases}
\frac{\partial u}{\partial \tau} + J_t(\frac{\partial u}{\partial t} - X_H(u)) = 0, \\
u(\tau, 0), \quad u(\tau, 1) \in L
\end{cases}
\end{align*}
and
\[E_J(u) \leq \lim E_K(u_\alpha) < \omega(A) + \|H\| < \infty.\]

By Lemma 11.2.2, we derive
\[L \cap \phi(L) \neq \emptyset.\]
This contradicts the assumption and hence finishes the proof. □

**Corollary 11.2.9** Consider a sequence of maps
\[u_\alpha \in \mathcal{M}_{K_\alpha}(J, H; A = 0)\]
such that
\[u_\alpha \to (u, v, w).\]
Then any bubble, if any, must have symplectic area less than $\|H\|$. 

**Proof** This is the case when $A = 0$. Then we have
\[E_{J_k}(u_\alpha) = \int \int \left\| \frac{\partial u_\alpha}{\partial \tau} \right\|_{J_k}^2 \leq \|H\|.
\]
Therefore,
\[E_{J_\omega}(u) + \sum_i \omega(v_i) + \sum_j \omega(w_j) \leq \|H\|.
\]
Since all terms are positive, this proves the corollary. □

**11.2.2 Proof of nondegeneracy**
We recall the geometric invariant $A(L; M, \omega)$ from Definition 8.4.10 which plays an important role in the study of the displacement energy of general compact Lagrangian submanifolds, especially of irrational Lagrangian submanifolds.

Let $J_0$ be any complex structure compatible with $\omega$. Consider
\begin{align*}
A_S(J_0, \omega) &= \inf_v \{\omega(v) \mid v : s^2 \to M \text{ non-constant, } \overline{\partial}_{J_0} v = 0\}, \\
A_D(J_0, \omega : L) &= \inf_w \{\omega(w) \mid w : (D^2, \partial D^2) \to (M, L) \text{ non-constant, } \overline{\partial}_{J_0} w = 0\}.
\end{align*}
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We have shown that $A_S(J_0, \omega), A_D(J_0, \omega; L) > 0$. Denote

$$A(J_0, \omega; L) = \min\{A_D(J_0, \omega; L), A_S(J_0, \omega)\}.$$ 

Then we take the supremum thereof over $J_0 \in \mathcal{J}_\omega$:

$$A(L; M, \omega) = \sup_{J_0} A(J_0, \omega; L).$$

**Remark 11.2.10** The invariant $A(L; M, \omega)$ plays an important role in Chekanov’s proof (Che98) of the positivity of the displacement energy of general compact Lagrangian submanifolds, especially of irrational Lagrangian submanifolds. If $\omega$ is weakly exact, i.e., if $\omega|_{\pi_2(M)} = 0$, we set $A(J_0, \omega) = \infty$ for any $J_0$. We also define $A_S = A_D = \infty$ if there exists no $J_0$-holomorphic sphere or disc. Furthermore, if $L \subset (M, \omega)$ is rational, we have $A(L; M, \omega) \geq \Gamma(L; M, \omega)$, where $\Gamma(L; M, \omega)$ is the positive generator of the period group of $L$

$$\{\omega(\beta) | \beta \in \pi_2(M, L)\}.$$ 

We note that for irrational $L \subset (M, \omega)$ the period group is dense in $\mathbb{R}$ and so there is no such topological lower bound for the displacement energy.

The following proof is borrowed from that of (Oh97c), which extracts the essential component of Chekanov’s proof just for the purpose of obtaining positivity of the displacement energy. Chekanov’s proof proves the stronger result in that it also estimates the number of intersections when $L$ and $\phi(L)$ intersect.

**Theorem 11.2.11** (Chekanov) *Let $(M, \omega)$ be a tame symplectic manifold and $L \subset (M, \omega)$ be a compact Lagrangian embedding. Then we have*

$$e(L; M, \omega) \geq A(L; M, \omega).$$

**Proof** If $L$ is not displaceable, by definition, we have $e(L; M, \omega) = \infty$. Then there is nothing to prove.

Now suppose there exists a Hamiltonian $H$ with $\phi_H^1(L) \cap L = \emptyset$. Let $H$ be any such Hamiltonian. We go back to the parameterized moduli space

$$\mathcal{M}^\text{para}_{[0, K_0]}(J, H; 0) = \bigcup_{K \in [0, K_0]} \{K\} \times \mathcal{M}_K(J, H; 0),$$

which is fibered over $[0, K_0]$. Consider the evaluation map

$$\text{Ev} : \mathcal{M}^\text{para}_{[0, K_0]}(J, H; 0) \to L \times [0, K_0]; \quad u \mapsto (u(0, 0), K).$$
11.2 Nondegeneracy of the Hofer norm

The transversality theorem implies that, for a generic choice of $J$, $\mathcal{M}^\text{para}_{[0,K_0]}(J,H;0)$ is a smooth manifold of dimension $(\dim L + 1)$ with boundary $\mathcal{M}_0(J,H,0) \bigsqcup \mathcal{M}_{K_0}(J,H,0)$.

We also know that $\text{ev}_0 : \mathcal{M}_0(J,H;0) \to L$ is a diffeomorphism.

In particular, the $\mathbb{Z}_2$-degree of $\text{ev}_0$ here is 1. On the other hand, we have $\mathcal{M}_{K_0}(J,H;0) = \phi$ by Proposition 11.2.7, hence the $\mathbb{Z}_2$-degree of the map $\text{ev}_{K_0}$ is zero. But the $\mathbb{Z}_2$-degree is invariant under a compact cobordism. Hence, $\mathcal{M}_{[0,K_0]}(J,H;0)$ cannot be compact! According to the Gromov–Floer compactness theorem, a bubble must develop, i.e., there exists a subsequence of $K_\alpha$ again denoted by the same symbol such that $K_\alpha \to K_\infty \in [0,K_0]$, and there must exist a sequence $u_\alpha \in \mathcal{M}_{K_\alpha}(J,H;0)$ converging to some cusp-curve $(u,v,w)$ with either $v \neq \emptyset$ or $w \neq \emptyset$. Recall that Corollary 11.2.9 implies that the symplectic energy of the bubble is always less than $\|H\|$, and hence

$$\min\{A_S(J(s,t),\omega), A_D(J_0,\omega;L)\} \leq \|H\|.$$ 

Now, it remains to compare $A_S(J(s,t),\omega)$ with $A_S(J_0,\omega)$. This can be done via the following upper semi-continuity result.

**Proposition 11.2.12** Suppose that $J_0$ is tame to $(M,\omega)$. Let $J = \{J(s,t)\}_{(s,t)\in[0,1]^2}$ be a family with $J_{|_{[0,1]^2}} \equiv J_0$ and $J_0 \equiv J$ outside a compact set of $M$. Then, for all $0 < \epsilon < A_S(J_0,\omega)$, we can choose $J : [0,1]^2 \to \mathcal{J}_\omega$ so that it is close to $J_0$ with

$$A_S(J(s,t),\omega) \geq A_S(J_0,\omega) - \epsilon \quad \text{for all} \quad (s,t) \in [0,1]^2.$$ 

**Exercise 11.2.13** Prove this proposition and show how the tameness of $J_0$ enters in the proof.

From this proposition, we obtain

$$\forall \epsilon > 0, \quad \min\{A_S(J_0,\omega) - \epsilon, A_D(J_0,\omega;L)\} \leq \|H\|.$$
Since this holds for all $\epsilon > 0$, we have

$$A(J_0, \omega; L) \leq \|H\|.$$  

By taking the supremum of $A(J_0, \omega; L)$ over $J_0$, we obtain $A(\omega, L) \leq \|H\|$ for any $H$ with $L \cap \phi_H^1(L) = \phi$. Now, taking the infimum of $\|H\|$ over all $H$ with $L \cap \phi_H^1(L) = \phi$, we obtain

$$0 < A(L; M, \omega) \leq \inf_H \{\|H\| \mid L \cap \phi_H^1(L) = \phi\}.$$  

But the quantity on the right-hand side of this inequality is nothing but the displacement energy $e(L; M, \omega)$ of $L$ and finishes the proof. \qed

Now we are ready to prove the nondegeneracy of the Hofer norm.

**Theorem 11.2.14** Let $(M, \omega)$ be a tame symplectic manifold and let $\phi \in \text{Ham}(M, \omega)$. Then $\phi = \text{id}$ if and only if $\|\phi\| = 0$.

**Proof** Obviously the zero Hamiltonian $H = 0$ generates the identity and hence the ‘only if’ part is trivial. Now we prove the ‘if’ part. Let $\phi \neq \text{id}$. Then there exists a point $x \in M$ with $\phi(x) \neq x$. By continuity we can find a small symplectic ball $B_\delta(x) \cong B^{2n}(\delta)$ centered at $x$ such that $\phi(B_\delta(x)) \cap B_\delta(x) = \emptyset$. Therefore $\phi$ displaces a compact Lagrangian torus $L \subset B_\delta(x)$, where $L = \phi(T_\epsilon^n)$ for

$$T_\epsilon^n = S^1(\epsilon) \times \cdots \times S^1(\epsilon) \subset B^{2n}(\delta) \subset \mathbb{C}^n$$

with $\epsilon$ sufficiently smaller than $\delta$ (in fact, it suffices to choose $\epsilon < \delta / \sqrt{n}$). Obviously we have

$$e(B_\delta(x)) \geq e(L) \geq A(L; M, \omega) > 0.$$  

But, since $\phi(B_\delta(x)) \cap B_\delta(x) = \emptyset$, we must have $\|\phi\| \geq e(B_\delta(x))$ by virtue of the definition of $e(B_\delta(x))$. On combining the two inequalities, we obtain $\|\phi\| > 0$, which finishes the proof. \qed

**Remark 11.2.15** We would like to mention that, in (LM95b), Lalonde and McDuff proved the above nondegeneracy on arbitrary symplectic manifolds by a different method. The tameness condition required in the proof of this section is an artifact of the analytic study of the perturbed Cauchy–Riemann equation (11.2.17), which requires the ambient manifold to be tame.
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